Random Lèvy matrices

Jerzy Jurkiewicz – Krakow
Mark Kac Complex Systems Research Centre, UJ

FENS Krakow 2006

Plan:

1. Concept of random matrices.
2. Random matrices - simplest example - Wigner matrices.
3. Random matrices - generalization - Bouchaud-Cizaux matrices.
4. Summary.

1. Random matrices

- Random matrices - a natural generalization of the concept of random numbers. Appear frequently in the description of massive data sets describing complex systems.
- Particularly interesting - realizations satisfying the condition of stability: the probability distribution is ”stable”, i.e. form-invariant under a convolution.
  In one dimension a convolution of two probability distributions leads to

$$P_{1+2}(X) = \int dX_1 P_1(X_1) \int dX_2 P_2(X_2) \delta\left(X - (X_1 + X_2)\right)$$

In this case stability means that for two arbitrary real constants $a,\ b$ we have

$$aX_1 + bX_2 \stackrel{D}{=} cX + d,$$

where X_1 and X_2 independent copies of a random variable and $\stackrel{D}{=}$ means equivalence in the distributional sense.

- Stable distributions play a special role - generalization of the Central Limit Theorem.
- We shall discuss the case where the $N \times N$ random matrix is very large: in fact we shall consider the possible existence of the limit $N \to \infty$.

2. Random matrices - simplest example - Wigner matrices

As a simple example of a random matrix we can consider a (symmetric) real $N \times N$ matrix $\mathbf{X}$ with elements $X_{ij},\ i \leq j$ independently drawn from the normal distribution. A probability to generate a particular matrix $\mathbf{X}$ is given by

$$P(\mathbf{X}) \sim \prod_{i \leq j} dX_{ij} \sim e^{-\frac{\beta}{2}\mathrm{tr}\mathbf{X}^2} D\mathbf{X}$$

where β is a scaling parameter, which has to be carefully chosen to guarantee the existence of the smooth limit for $N \to \infty$ (in this case $\beta \sim N$).
The object we shall study will be the distribution of eigenvalues of the random matrix $\mathbf{X}$

$$\rho(\lambda) = \langle \frac{1}{N} \sum_i \delta(\lambda_i - \lambda) \rangle$$

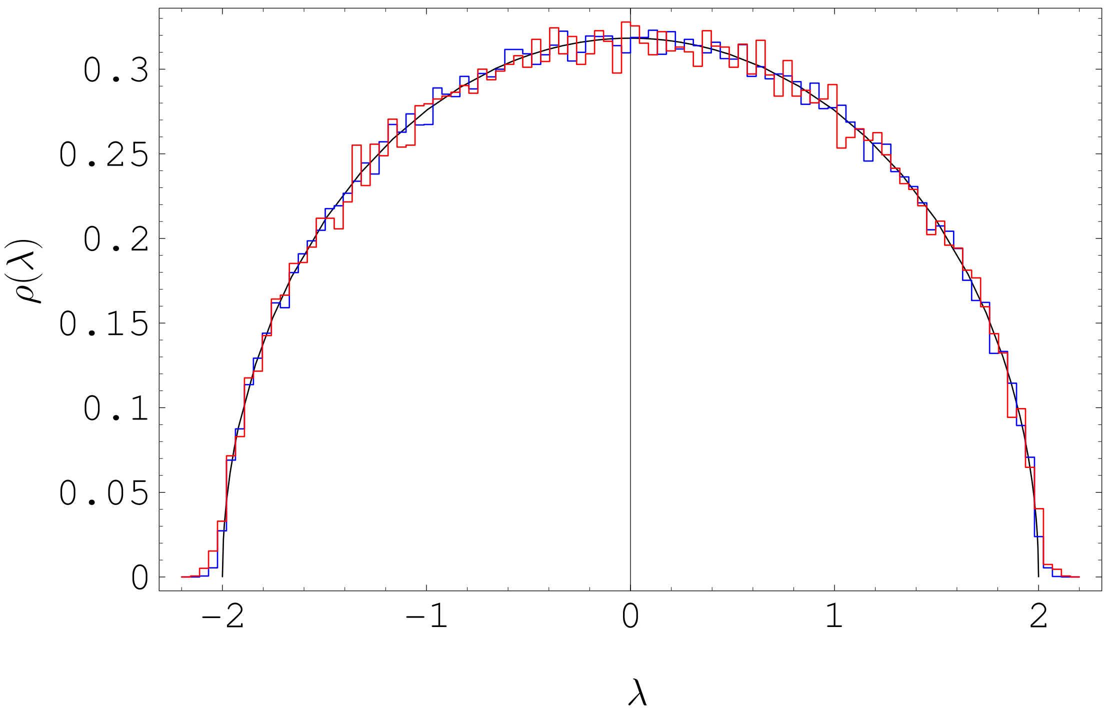
where the average is over the ensemble of all random matrices.

Properties:

- A sum of symmetric matrices is a symmetric matrix - we may define a matrix convolution in analogy with one-dimensional case.
- The probability measure is stable - it follows from the stability of individual elements of the matrix.
- For $N \to \infty$ the spectrum of such matrices ($\beta = N$) is given by the "Wigners semicircle"

$$\rho(\lambda) = \frac{1}{\pi}\sqrt{4-\lambda^2}, \quad -2 \leq \lambda \leq 2$$

Example to illustrate how well the $N \to \infty$ limit works for finite N ($N = 100,\ 200$)



3. Random matrices - generalization - Bouchaud-Cizaux matrices

A simple generalization proposed by Bouchaud and Cizeaux was to consider symmetric matrices $\mathbf{X}$ with elements $X_{ij},\ i \leq j$ generated from the Lèvy distribution.

Lèvy distributions - a family of one-dimensional stable distributions with power-like tail distribution.

$$L_{\mu}^{C,\beta}(x) \sim_{|x| \to \pm\infty} \frac{C(1 \pm \beta)}{|x|^{1+\mu}}$$

characterized by:

- index $0 < \mu \leq 2$ ($\mu = 2$ is a normal distribution)
- asymmetry parameter $-1 \leq \beta \leq 1$
- scale parameter $C > 0$

B-C matrix measure **is not** rotationally symmetric (except $\mu = 2$).

Spectrum of B-C matrices in the large-N limit requires $C \sim N^{1/\mu}$.
Explicit solution can be obtained:

$$\rho(\lambda) = L_{\mu/2}^{C(\lambda),\beta(\lambda)}(\lambda)$$

with additional self-consistency constraints

$$C(z) = \int_{-\infty}^{\infty} \frac{dG}{G^2} |G|^{\mu/2} L_{\mu/2}^{C(z),\beta(z)}(z - 1/G)$$

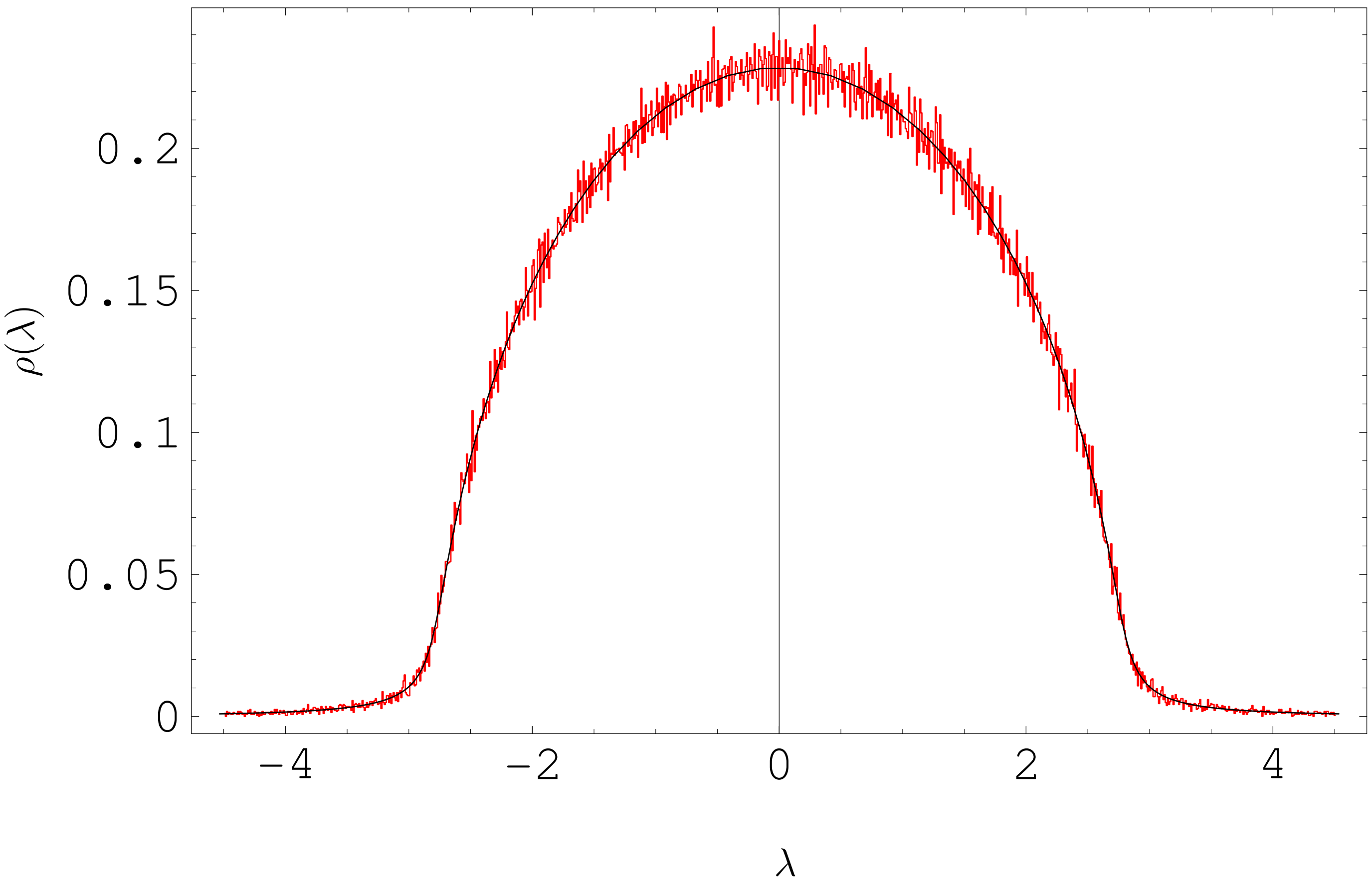
and

$$\beta(z) = \frac{\int_{-\infty}^{\infty} \frac{dG}{G^2} |G|^{\mu/2} \mathrm{sgn}(G) L_{\mu/2}^{C(z),\beta(z)}(z - 1/G)}{\int_{-\infty}^{\infty} \frac{dG}{G^2} |G|^{\mu/2} L_{\mu/2}^{C(z),\beta(z)}(z - 1/G)}$$

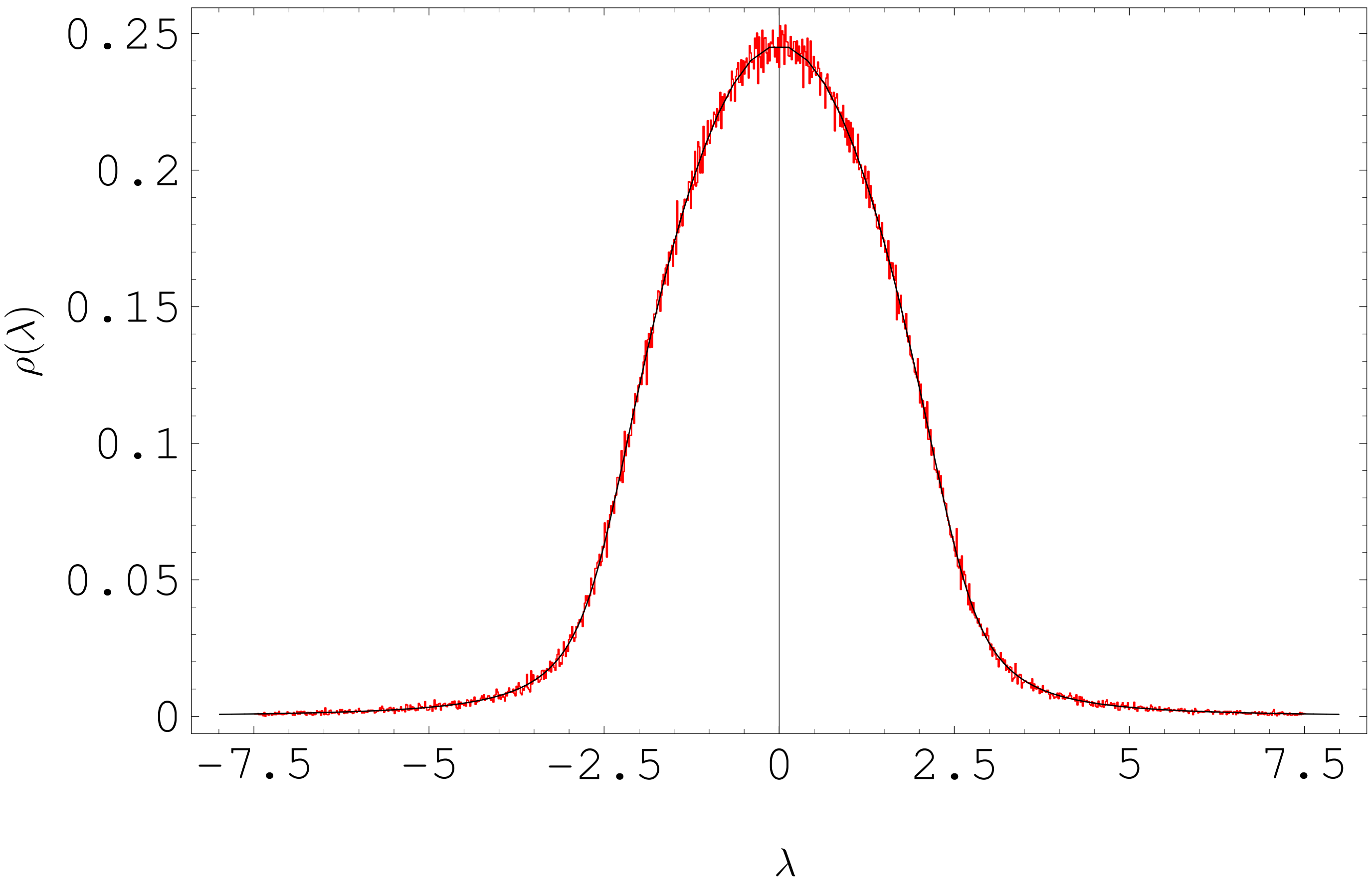
Except for few cases functions $L_{\mu}^{C,\beta}(x)$ are not expressible by elementary functions, but can be calculated numerically.

Properties of the spectrum:

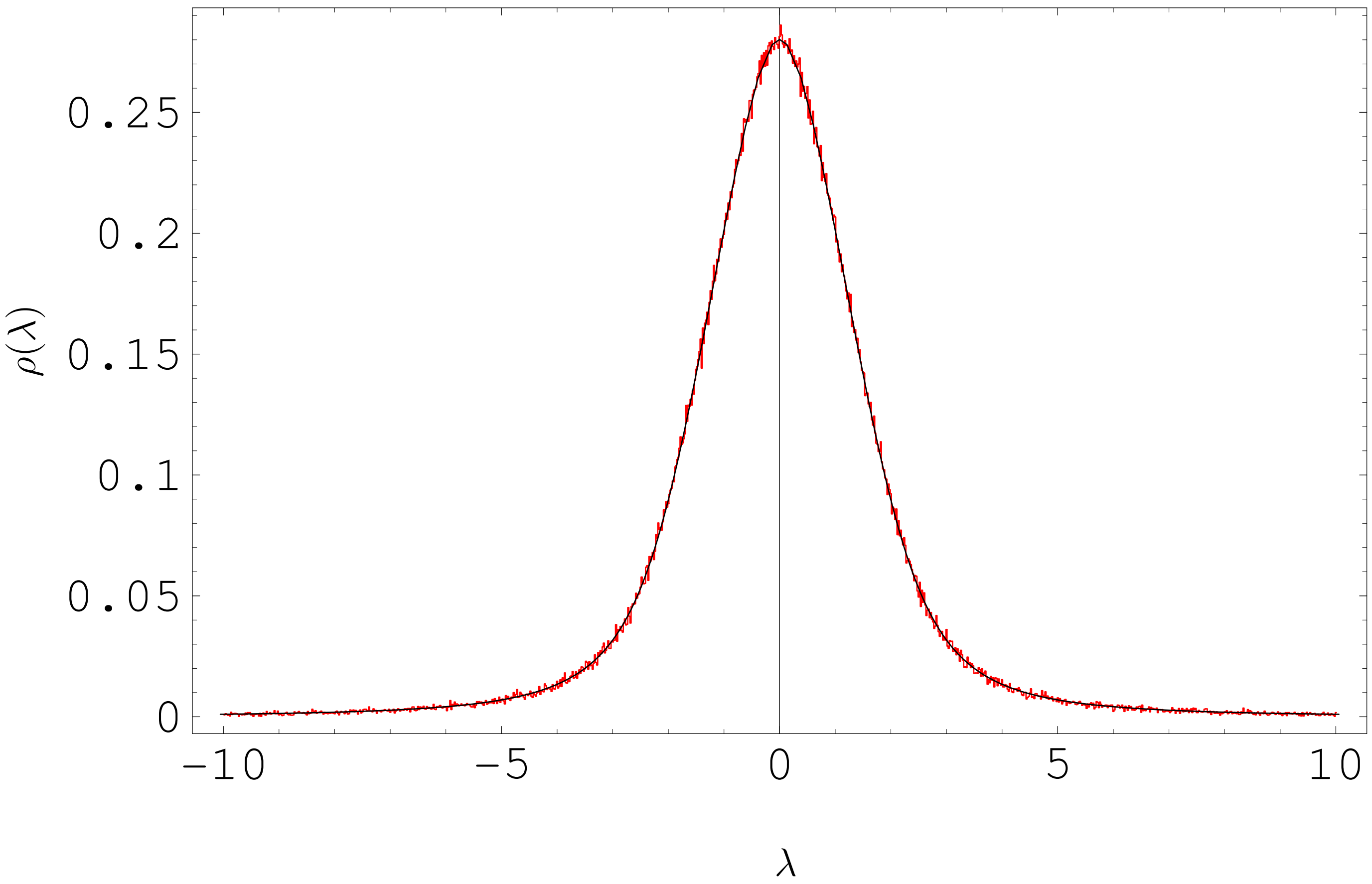
- Has a power tail $\rho(\lambda) \sim 1/|\lambda|^{1+\mu}$ fpr $\lambda \to \pm\infty$
- Does not depend on asymmetry parameter β of the individual element distribution
- Small finite-N corrections (see below comparison for $N = 400$).



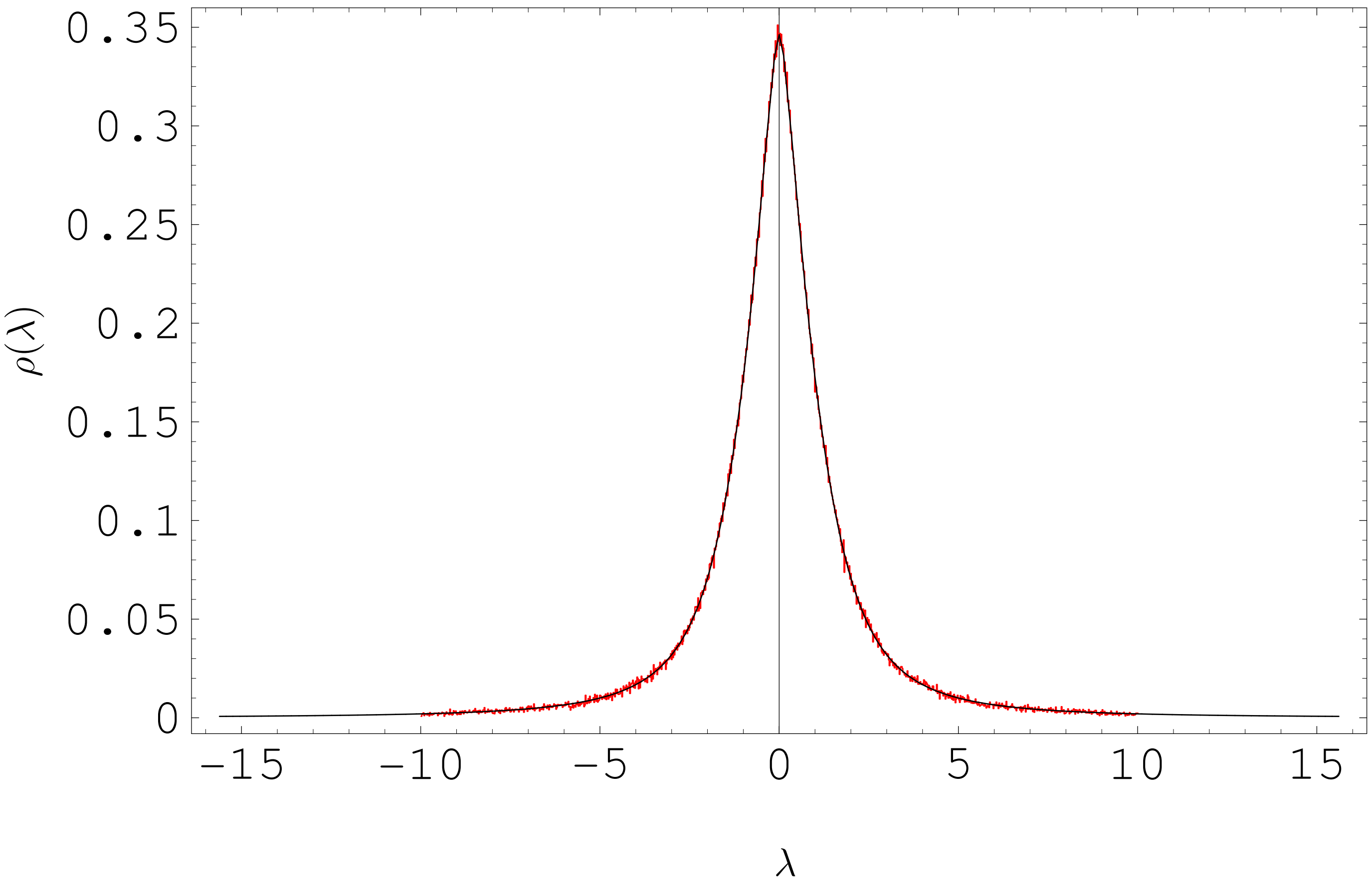
Theoretical (black) and numerical (red) eigenvalue distributions for $\mu = 1.95$.



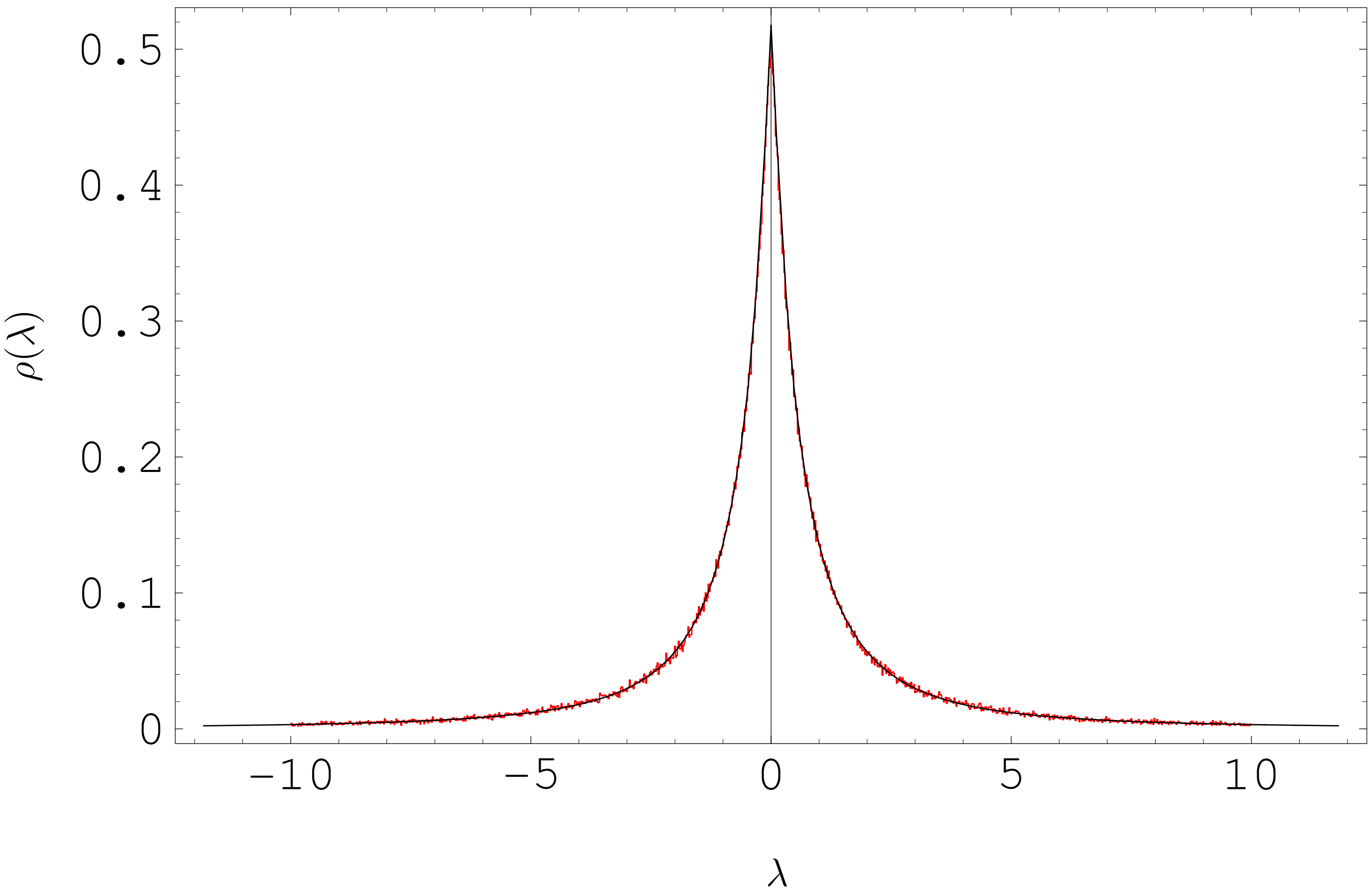
Theoretical (black) and numerical (red) eigenvalue distribution for $\mu = 1.75$



Theoretical (black) and numerical (red) eigenvalue distribution for $\mu = 1.50$



Theoretical (black) and numerical (red) eigenvalue distribution for $\mu = 1.25$



Theoretical (black) and numerical (red) eigenvalue distribution for $\mu = 1.00$

4. Summary

Large class of random matrices which have a relatively simple analytic spectral description for large-N limit.
Power-like spectra appear in financial data. Random matrices are a first step to study possible correlations in a massive random data.
Lèvy-type matrices require a completely new concept of correlations - work in progress.
Relation to FRV requires introducing the rotational symmetry.

Analogy to the Central Limit Theorem for matrices.
$A_i, \ i = 1, \dots \mathcal{N}$ - B-C matrices.
$O_i, \ i = 1, \dots \mathcal{N}$ - random orthogonal matrices.

$$B = \frac{1}{\mathcal{N}^{1/\mu}} \sum_i^{\mathcal{N}} O_i A_i O_i^T$$

B matrices have the FRV spectra for $\mathcal{N} \to \infty$.