Dynamics of the Warsaw Stock Exchange index as analysed by the Mittag-Leffler function

Marzena Kozłowska and Ryszard Kutner

Department of Physics, Warsaw University

Physics of Socio-Economic Systems
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Dynamics of the Warsaw Stock Exchange index as analysed by the fractional relaxation equation

Marzena Kozłowska, Ryszard Kutner

Division of Physics Education

Institute of Experimental Physics

Department of Physics, Warsaw University

2 Ogólnopolskie Sympozjum z Fizyki w Ekonomii i Naukach Społecznych

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Schedule

- Motivation and inspiration
- Model: fractional risings and relaxations decorated by oscillations
- Comparison with empirical data
- Microscopic model: hierarchical network of investors
- Conclusions

Motivation and inspiration

WIG is the oldest index of the WSE.
Duration of its peaks covers ~3/4 time-range.
Definition of WIG analogous to S&P500.
Our aim is to describe:

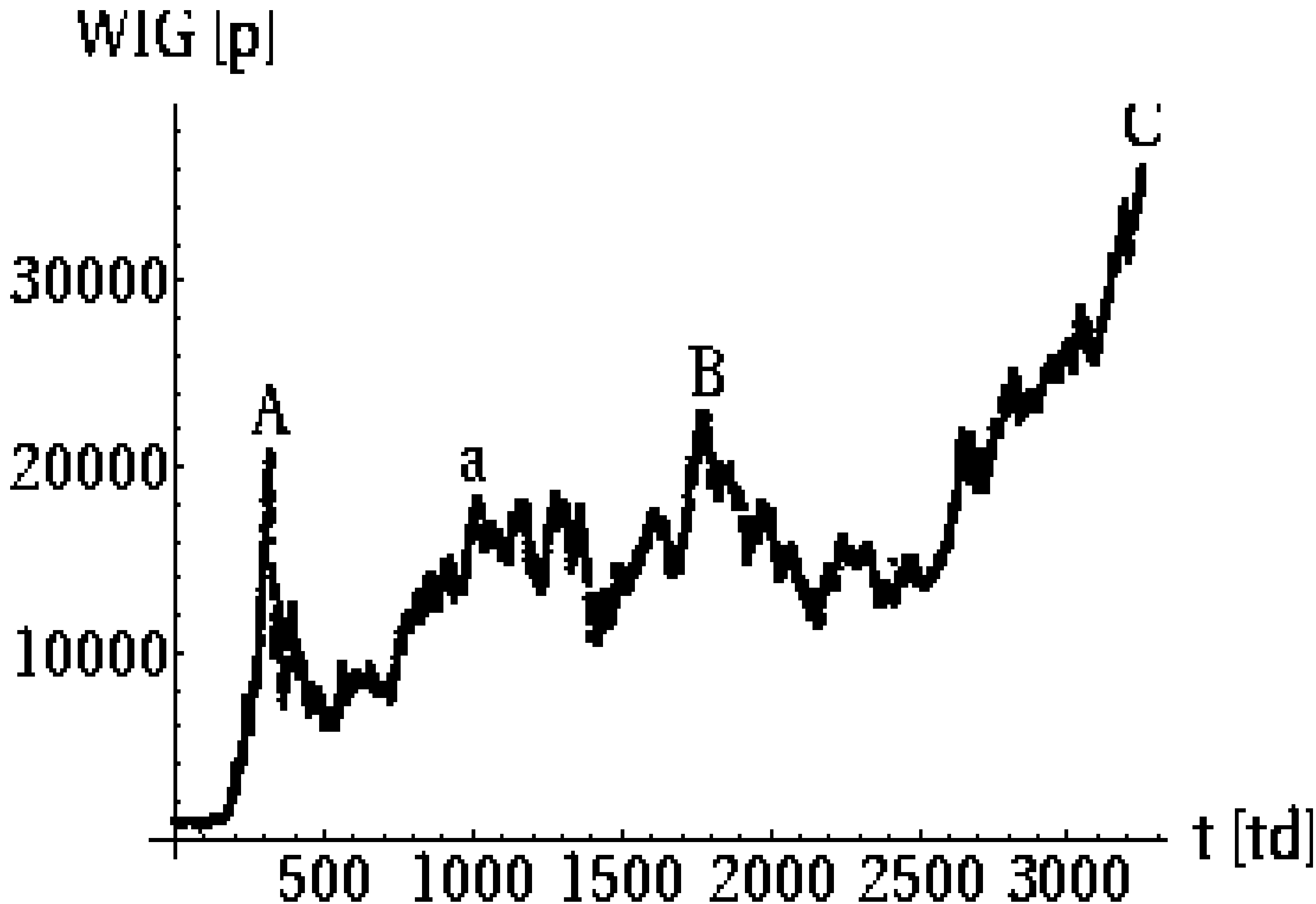
- Slowing down of risings and relaxations within intermediate (and possibly long) time-ranges of temporal maximums of WIG.
- Systematic oscillations of WIG within maximums.
- Possibly sudden increase of market activity in the vicinity of temporal maximums of WIG.
- Retarded feedback of WIG (a kind of memory): main effect responsible for temporal maximums.

Model: fractional risings, relaxations and oscillations.

Model consists of two steps

- Linear, ordinary differential equation of the first order without any retardation describing evolution of a hypothetical auxiliary index.
- Glöckle-Nonnenmacher conjecture which transforms this equation to a more general fractional differential one which already desribes evolution of an empirical index. This was done in analogy to similar step performed for viscoelastic materials.

WIG dzienny na zamknięciu: 16.04.91 - 31.12.05



Definitions of indexes

Defintion of WIG and S&P500

WIG(t), *S&P500(t)* ~ $K(t)\sum L_j(t)X_j(t)$

$K(t)$: correction

$L_j(t)$: current number of stocks of *j*-th company on stock market

$X_j(t)$: price of stock of *j*-th company

Model: fractional risings, relaxations and oscillations

Instantaneous decomposition of *WIG(t)*

$WIG(t) = A\,U(t) + B\,V(t) \quad (\geq 0)$

U(t): instantaneous offset between demand and supply

V(t): volume trade

A, B, C, D, E: coefficients

Instantaneous dynamics

$$\frac{dV(t)}{dt} = C\ V(t) + D\ WIG(t) + E\ \frac{dWIG(t)}{dt}$$

Model: fractional risings, relaxations and oscillations

Conveniet equation without *V(t)*:

$$\frac{dWIG(t)}{dt} = -\frac{1}{\tau} WIG(t) \pm \frac{A'}{\tau_0} U(t) + A' \frac{dU(t)}{dt}$$

$$A' = \frac{A}{1 - BE}, \quad \tau_0 = \frac{1}{(C)}, \quad \tau = \left(\frac{C + BD}{1 - BE}\right)^{-1}$$

Conjecture: usual differentiations are replaced by fractional differentiations

↓

Model: fractional risings, relaxations and oscillations

Dynamics of WIG is described by the linear fractional differential equation.

Free case U=0 (for $0 < \alpha \leq 1$):

$$\frac{dWIG(t)}{dt} = -\frac{1}{\tau^{\alpha}}\, {}_0D_t^{1-\alpha} WIG(t), \quad 0 < \alpha \leq 1$$

$${}_0D_t^{1-\alpha} WIG(t) = \frac{d}{dt}\, {}_0D_t^{-\alpha} WIG(t)$$

$${}_0D_t^{-\alpha} WIG(t) = \frac{1}{\Gamma(\alpha)} \int \frac{WIG(y)}{(t-y)^{1-\alpha}} dy$$

Retarded feedback well seen

Free solution

$$WIG(t) = WIG(0)\,E_{\alpha}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right), \quad 0 < \alpha \leq 1$$

Mittag-Leffler fnction (generalised exponent)

$$E_{\alpha}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right) = \sum \frac{\left(-(t/\tau)^{\alpha}\right)^{n}}{\Gamma(1+\alpha n)}$$

Characteristic limits of Mittag-Leffler function

Stretched exponential function or Kohlrausch-Williams-Watts (KWW) decay:

$$E_{\alpha}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right) \approx \exp\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right), \quad 0<\alpha\leq 1, \quad t \ll \tau$$

Power-law or Nutting law:

$$E_{\alpha}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right) \approx \frac{1}{\Gamma\left(1-\alpha\right)} \frac{1}{\left(t/\tau\right)^{\alpha}}, \quad 0<\alpha\leq 1, \quad t \gg \tau$$

Model: fractional risings, relaxations and oscillations

Dynamics of WIG is described by the linear fractional differentia equation.

Full case (for $0 < \alpha \leq 1$):

$$\frac{dWIG(t)}{dt} = -\frac{1}{\tau^{\alpha}} {}_0D_t^{1-\alpha} WIG(t) \mp \frac{A'}{\tau_0^{\alpha}} {}_0D_t^{1-\alpha} U(t) + A' \frac{dU(t)}{dt}$$

U(t) – instantaneous offset between demand and supply.

For example, a wiggle:

$$U(t) \sim \exp\left(i\left(\omega - \Delta\omega\right)t\right) + \exp\left(i\left(\omega + \Delta\omega\right)t\right)$$

Approximate solution

$$WIG\left(\left(t - t_{MAX}\right)\right) \sim E_{\alpha}\left(-\left(\frac{\left(t - t_{MAX}\right)}{\tau}\right)^{\alpha}\right) + C\cos\left(\omega\left(t - t_{MAX}\right)\right)\cos\left(\Delta\omega\left(t - t_{MAX}\right)\right)$$

Comparison with empirical data gives:

$0.25 \leq \alpha \leq 0.42$

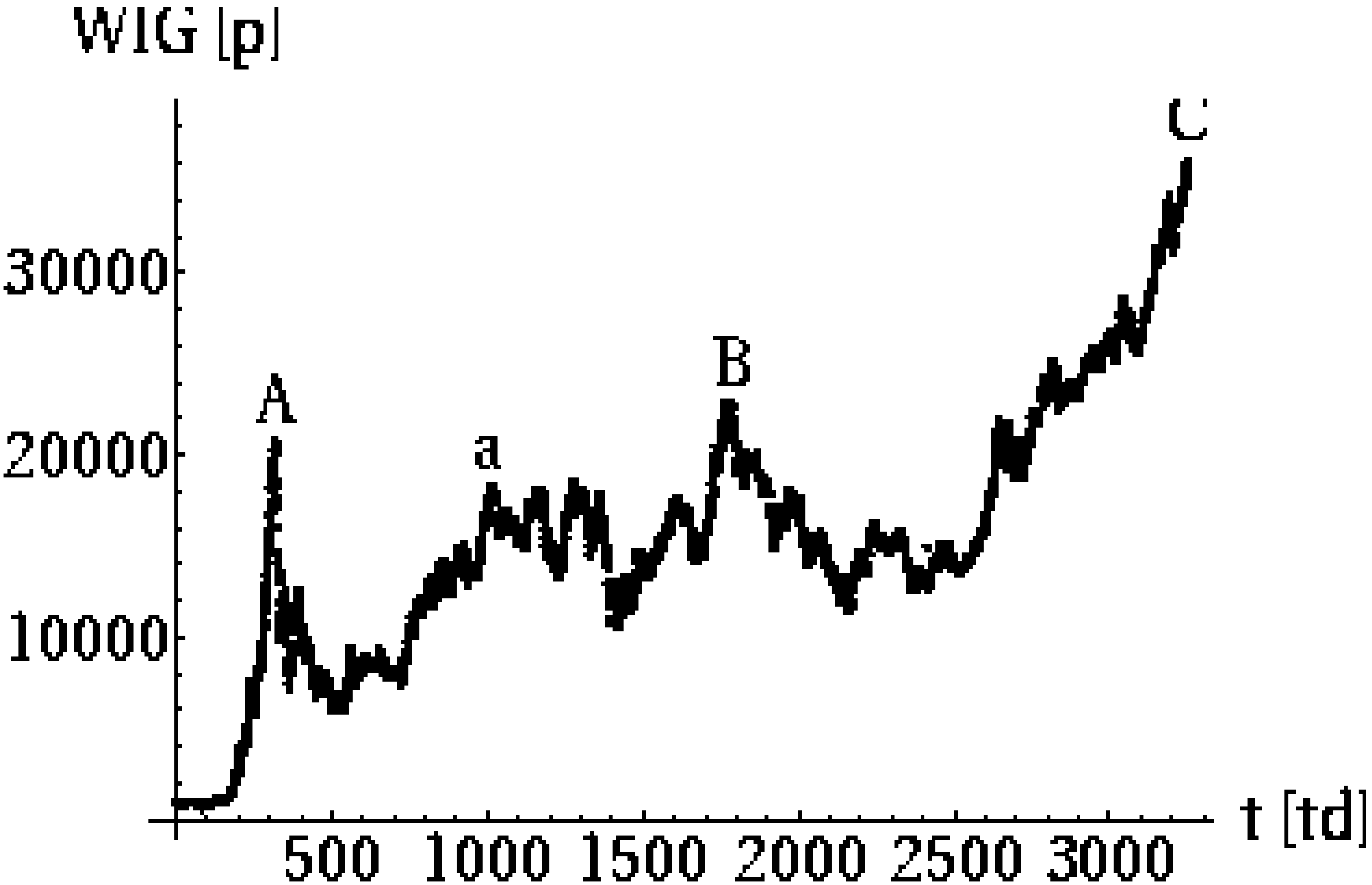
$60 \leq \tau \leq 320$

$0.1 \leq \omega \leq 0.2$

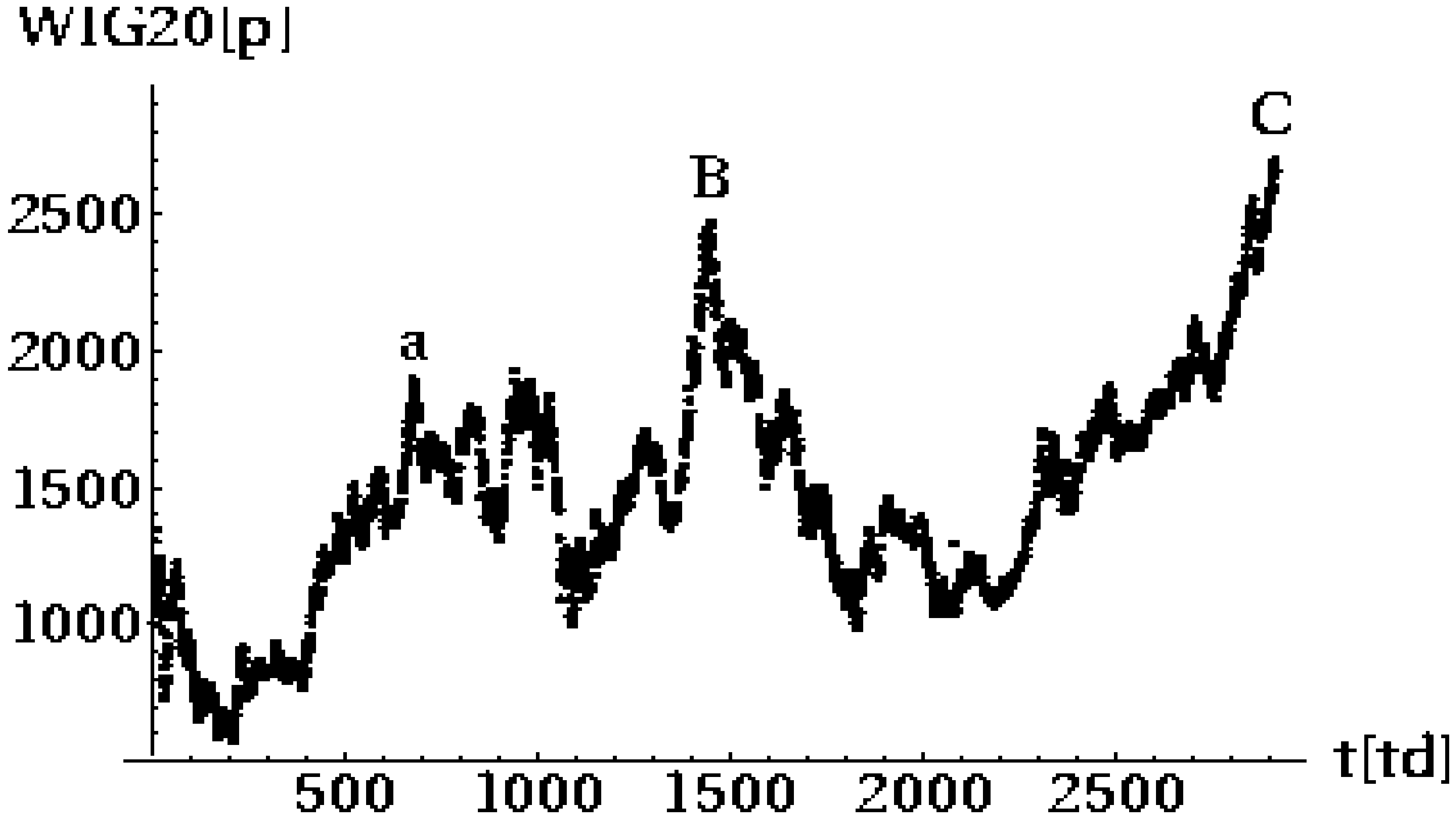
$0.01 \leq \Delta\omega \leq 0.02$

$1500 \leq |C| \leq 4500$

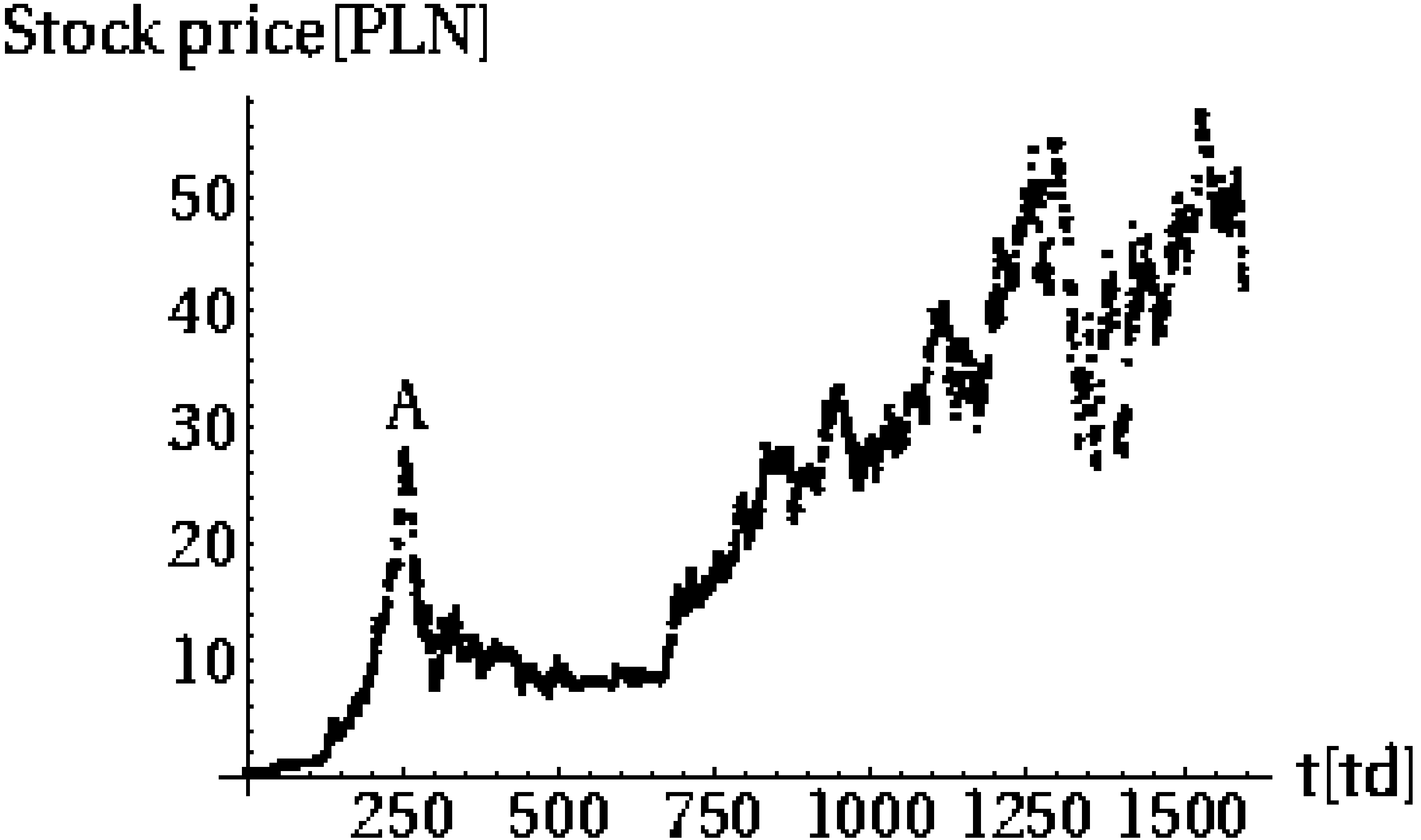
Daily closing price of WIG



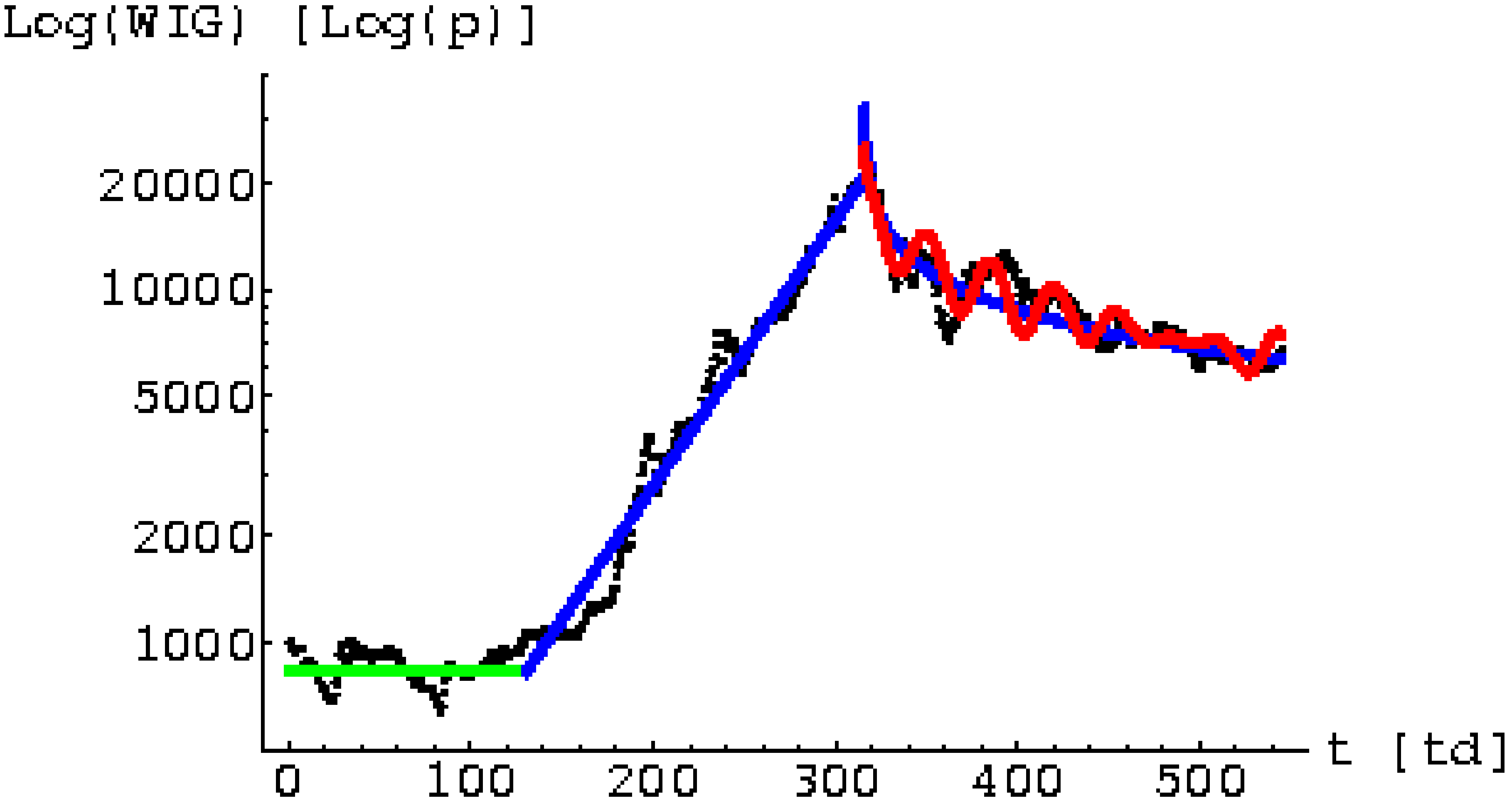
Daily closing price of WIG20



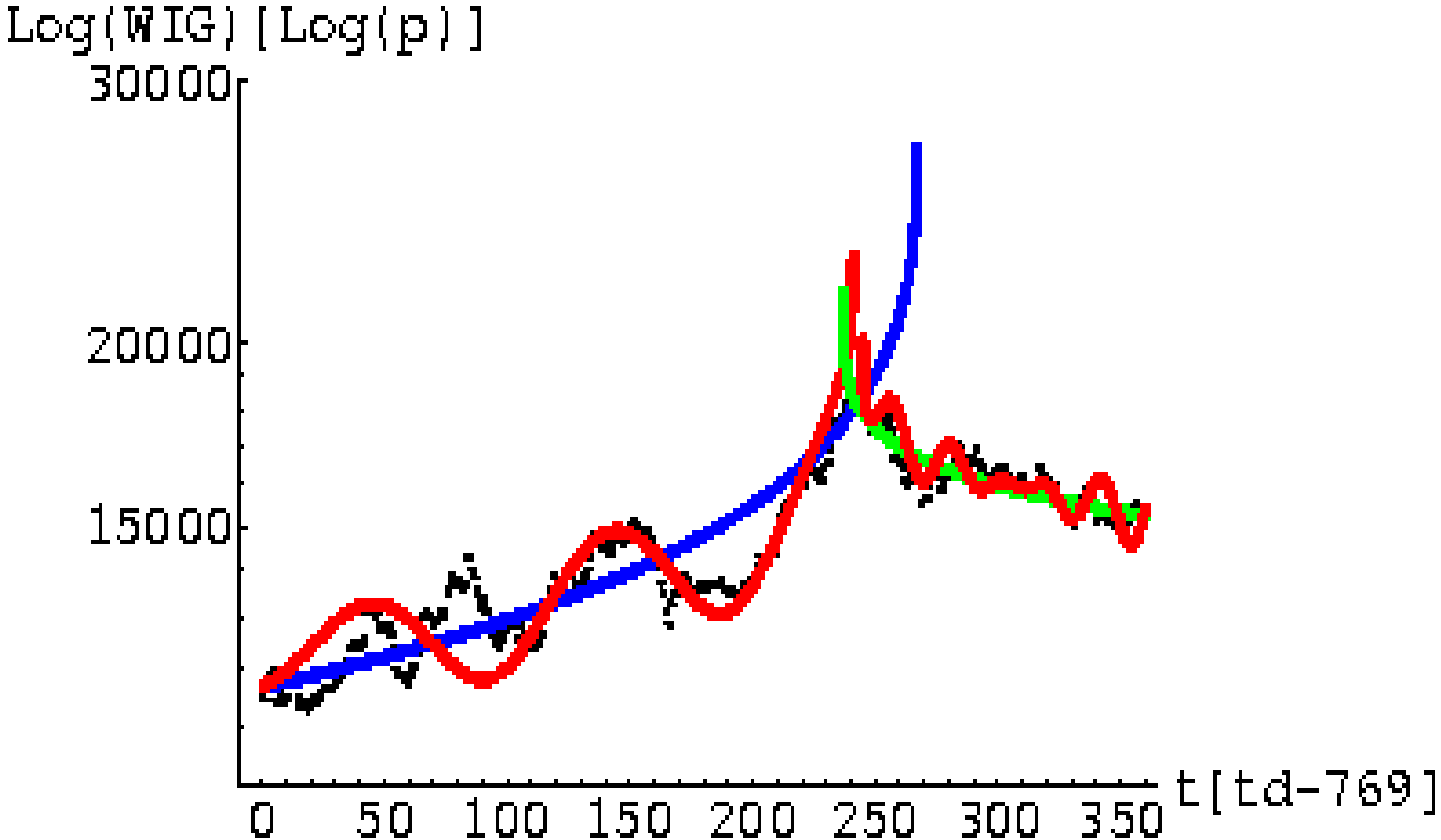
Daily closing price of Elektrim-Vivendi Company



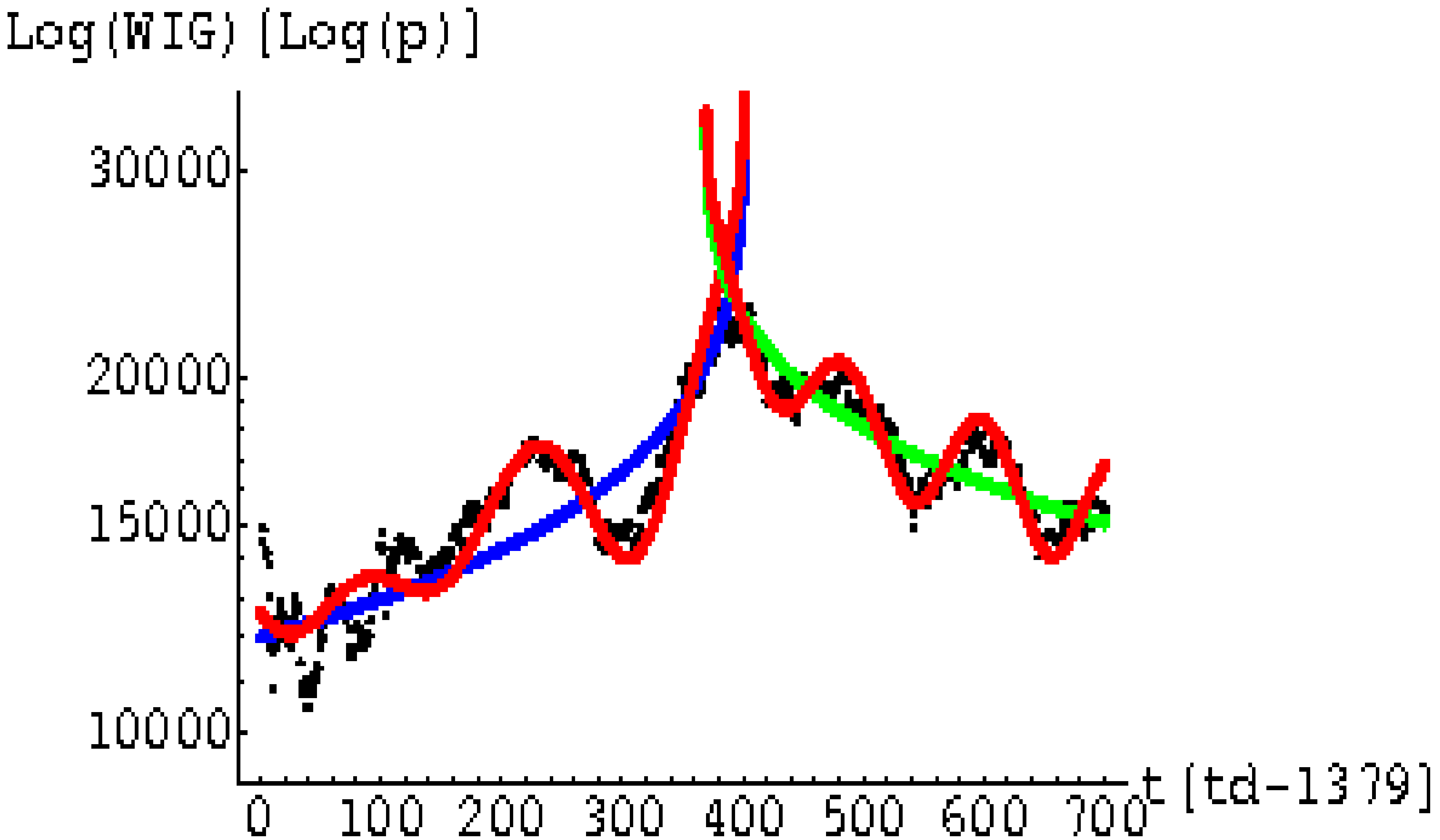
Temporal maximum A of WIG



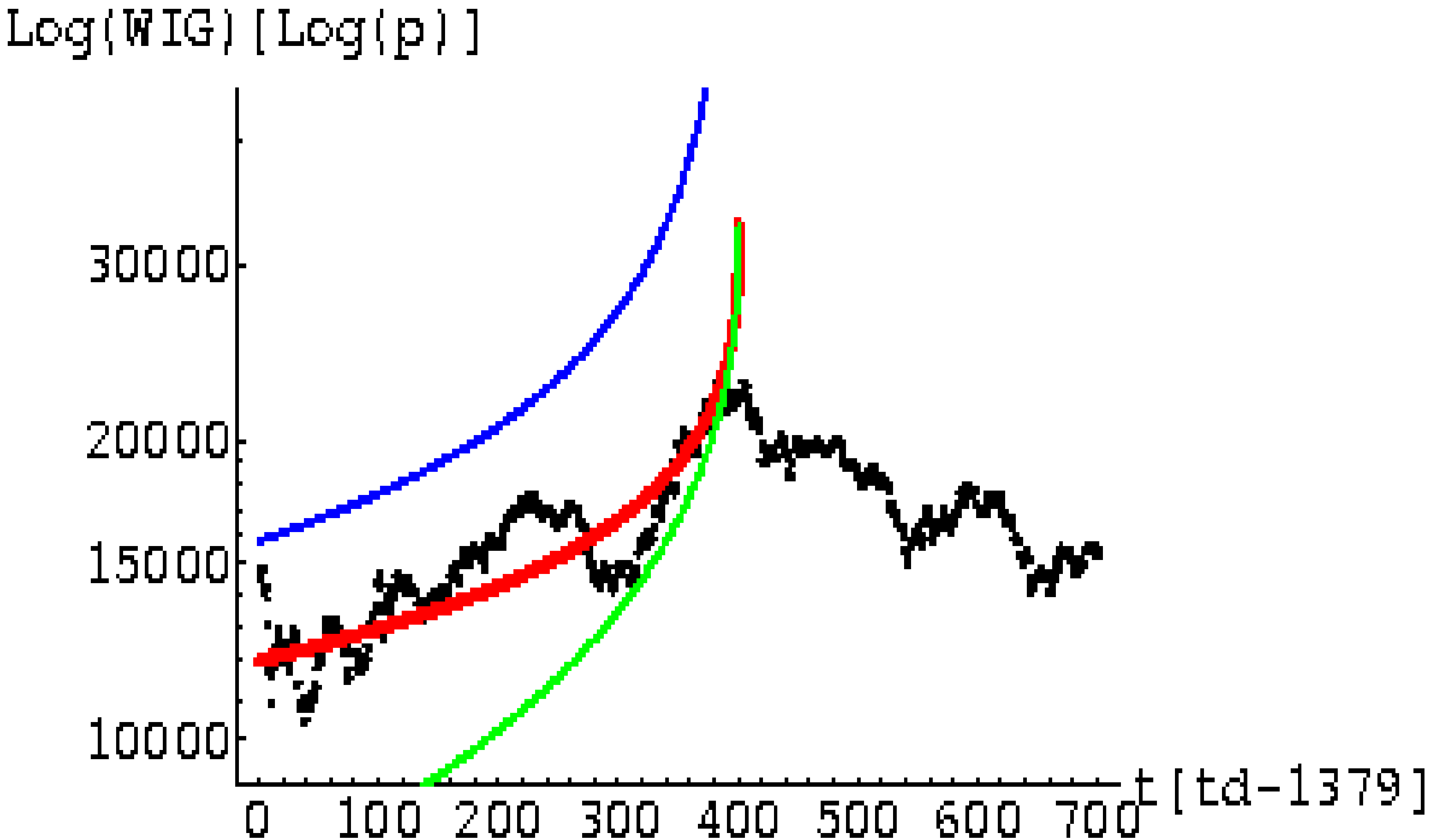
Temporal maximum a of WIG



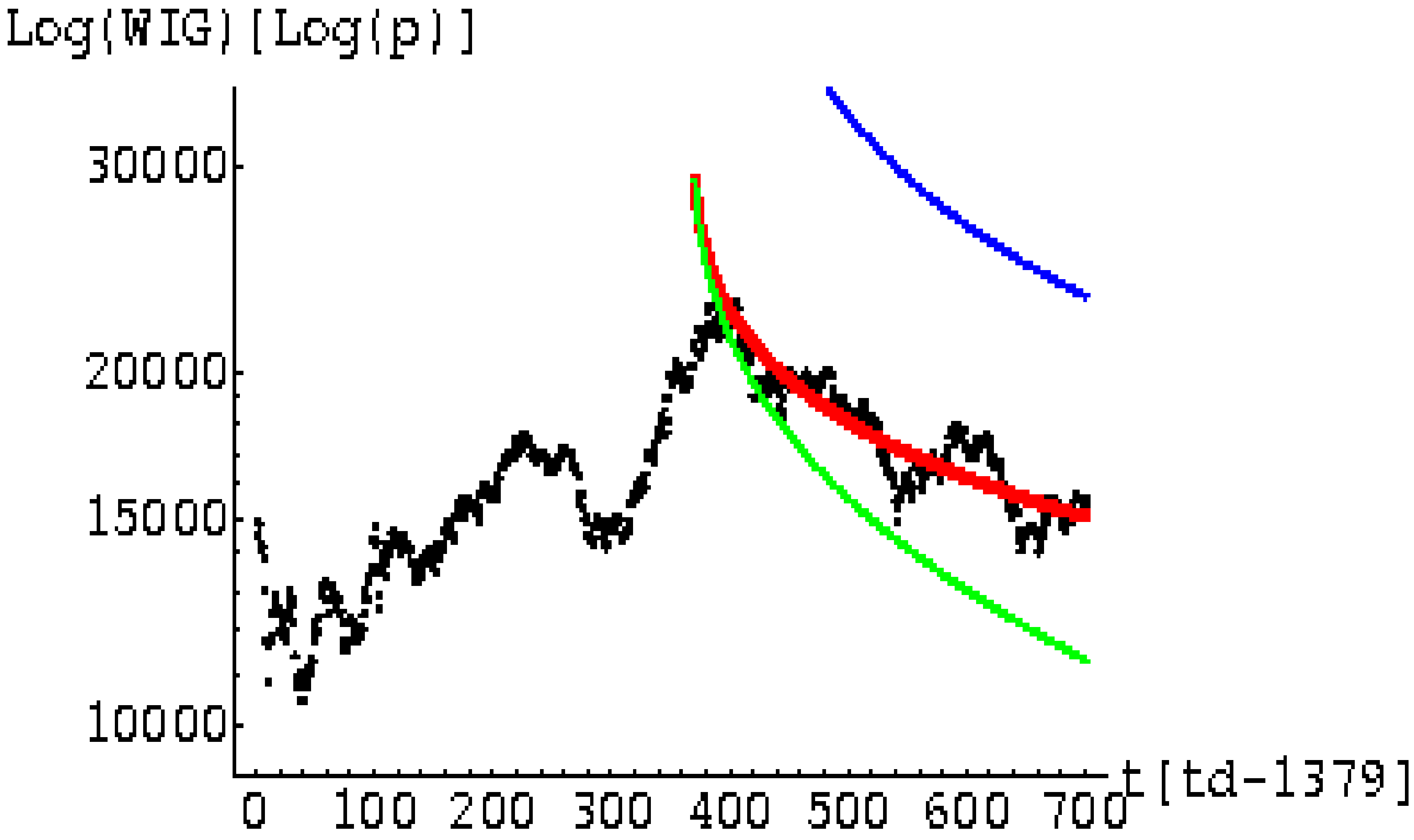
Temporal maximum B of WIG



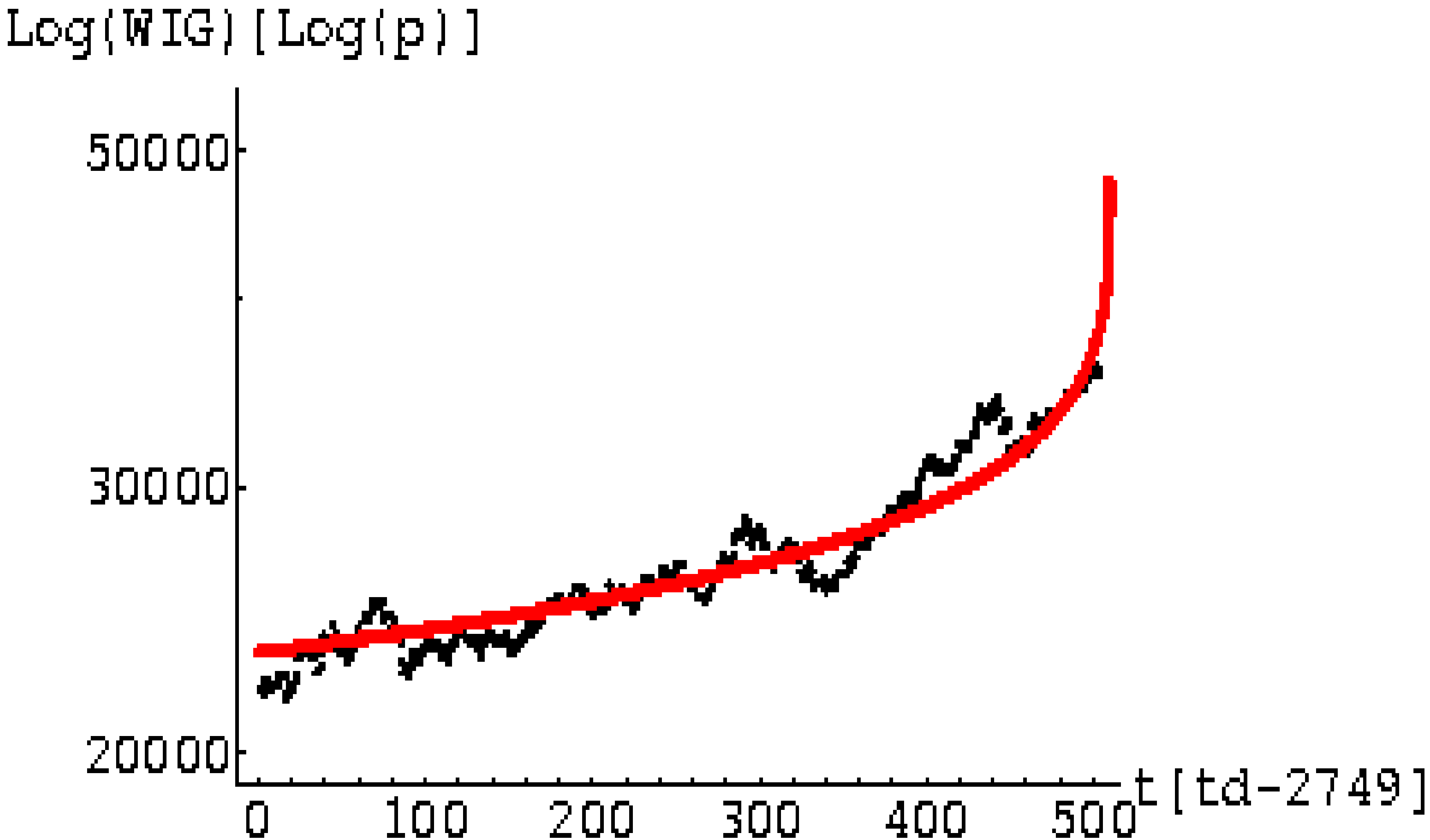
Maximum B: KWW & N limits of Mittag-Leffler function



Maximum B: KWW & N limits of Mittag-Leffler function



Temporal maximum C of WIG



Model mikroskopowy: hierarchia inwestorów

Tabela analogonów

wydłużenie → $\varepsilon \equiv X$ ← wartość indeksu

naprężenie → $\sigma \equiv u$ ← różnica pomiędzy popytem a podażą

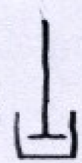
$\varepsilon^s \equiv X^s$

$\varepsilon^d \equiv X^d$

$\sigma^s \equiv u^s$

$\sigma^d \equiv u^d$

E, E_j  $\equiv$ chęć do aktywności

η, η_j  $\equiv$ obawa przed aktywnością

- $\varepsilon_j^d = \varepsilon_{j+1}^s + \varepsilon_{j+1}^d \, , \quad j = 0, 1, \dots, n-2$ ← wydłużenie
- $\varepsilon_{n-1}^d = \varepsilon_n^s$
- $\varepsilon = \varepsilon_0^s + \varepsilon_0^d$

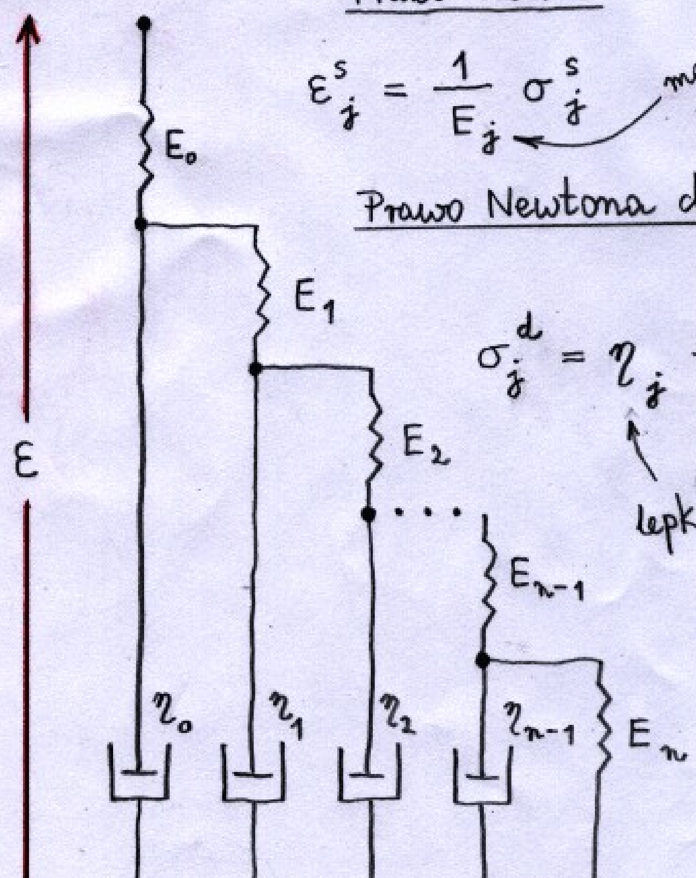
naprężenie →

- $\sigma_j^s = \sigma_j^d + \sigma_{j+1}^s \, , \quad j = 0, 1, \dots, n-1$
- $\sigma = \sigma_0^s$

Warunki początkowe

$\varepsilon_j^d(0) = 0,$

$j = 1, 2, \dots, n-1$



Prawo Hooka

$$\varepsilon_j^s = \frac{1}{E_j} \sigma_j^s$$

E_j ← moduł Younga

Prawo Newtona dla płynów

$$\sigma_j^d = \eta_j \frac{d\varepsilon_j^d}{dt}$$

η_j ← lepkość

Wynik

- $$\frac{d\,\varepsilon(t)}{dt} = \frac{1}{\eta_0^{\alpha}\, E_0^{1-\alpha}}\; {}_0D_t^{1-\alpha}\, \sigma(t)$$

Analogon giełdowy

- $$\frac{d}{dt} X(t) = \frac{A'}{\xi}\left[\pm\frac{1}{\tau_0^{\alpha}} + \frac{1}{(1-\xi)\,\tau^{\alpha}}\right] {}_0D_t^{1-\alpha}\, u(t)$$

- $$X(t)\,\frac{1-\xi}{A'} = u(t)$$

↑ współczynnik przełożenia

<u>Uwaga</u>: nasze wyjściowe równanie oraz równanie drugie dają równanie pierwsze.

Parametryzacja hierarchii

$$\frac{E_1}{E_0} = 1 - \alpha \quad , \quad 0 < \alpha < 1$$

$$\frac{\eta_0}{\eta_1} = 2\,\frac{1-\alpha}{\alpha}$$

parzyste $n \geq 4$

$$\frac{E_{n-2}}{E_{n-3}} = \frac{n-3}{n-1}\,\frac{\frac{n}{2} - \alpha}{\frac{n}{2} - 2 + \alpha}$$

nieparzyste $n \geq 5$

$$\frac{E_{n-2}}{E_{n-3}} \cdot \frac{\eta_{\frac{n-1}{2}-1}}{\eta_{\frac{n-1}{2}}} = \frac{n-3}{n-1} \cdot \frac{\frac{n-1}{2}}{\frac{n-1}{2} - 1} \cdot \frac{\frac{n-1}{2} - 1 + \alpha}{\frac{n-1}{2} - \alpha}$$

Podsumowanie

- Praktycznie rzecz biorąc, w obrębie maksimów inwestorzy stanowią układy (sieci) pośrednie pomiędzy opisywanymi prawem KWW a prawem Nuttinga.

- Maksima dobrze opisuje się funkcjami Mittag-Lefflera udekorowanymi różnego rodzaju oscylacjami.

- Są one rozwiązaniem ułamkowych równań różniczkowych, za którymi stoi przede wszystkim efekt opóźnionego sprzężenia zwrotnego w dynamice indeksu.

- Równania te można otrzymać z poziomu mikro zakładając hierarchiczną strukturę powiązań pomiędzy inwestorami.

Podsumowanie c.d.

- **Teoretyczna szybkość** $\frac{dX(t)}{dt}$ **rozbiega się jak** $\frac{1}{|t-t_{MAX}|^{1-\alpha}}$ **gdy** $t \to t_{MAX}$ **z lewej lub prawej strony. Przypuszczamy, że mamy tutaj do czynienia z dynamicznym przejściem fazowym (faza wzrostu – faza spadku).**

- **Podejście można stosować do innych indeksów a także cen akcji.**