RESPONSE TO LÉVY NOISE
AND FLUCTUATION-DISSIPATION
RELATION

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Seminarium Wydziału Fizyki i Informatyki Stosowanej, AGH, 26.10.2012
Motivation

• Lévy statistics: implies new properties departing strongly (quantitatively and qualitatively) from standard statistical behaviors

• Have been addressed in various fields: dynamics in plasma, self-diffusion in micelle systems, exciton and charge transport in random polymers under conformational motion, laser cooling and coherent radiation trapping, analysis of complex systems...

• Nonequilibrium noises - non-Gaussian character, in general, space and time correlated

• Thermodynamics in the presence of non-Gaussian fluctuations?
Wiener process and Brownian motion

- Wiener process $\mathcal{W}(t)$: stationary with independent and Gaussian-distributed increments, $\langle \mathcal{W}(t)\mathcal{W}(s) \rangle = \min(t, s)$
- Representation of a Brownian motion: a limit in distribution of i.i.d (Gaussian) jumps taken at infinitesimally short time intervals of random length $\mathcal{W}(t) = \lim_{n \to \infty} \sum_{k=1}^{N(nt)} X_k$

\[
\frac{\partial p(x, t)}{\partial t} = \sigma^2 \frac{\partial^2 p(x, t)}{\partial x^2}
\]

\[
m \ddot{x} + U'(x) + \eta \dot{x} = \xi(t)
\]

\[
\langle \xi(t)\xi(t') \rangle = 2\Gamma \delta(t - t'),\: \langle \xi(t) \rangle = 0
\]

\[
\Gamma = \eta k_B T,\: \langle x^2(t) \rangle \xrightarrow{t \to \infty} \frac{2k_B T}{m\eta} t
\]

\[
\langle x^2 \rangle = 2\sigma^2 t
\]

strength of fluctuations related to the magnitude of dissipation
Scaling laws:

complex systems - time/space  
series analysis

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**pink noise**

\[ S(f) = const \times f^{-\alpha} \]
Non-Gaussian stable white noise

- Generalized Wiener process $W_{\alpha,\beta}(t)$ – non-Gaussian, with stationary and independent increments distributed according to the $\alpha$-stable law

$$W_{\alpha,\beta}(t) = \int_0^t \zeta(s) ds = \int_0^t dL_{\alpha,\beta}(s) \approx \sum_{i=0}^{N-1} (\Delta s)^{1/\alpha} \zeta_i,$$

$\zeta_i$: i.i.d variables with the stable Lévy probability density function $l_{\alpha,\beta}(\zeta)$, $N\Delta s = t - t_0$, asymptotics $l_{\alpha,\beta}(\zeta) \leq |\zeta|^{-1-\alpha}$.

$$\phi_\zeta(k) = \int_{-\infty}^{+\infty} d\zeta e^{ik\zeta} l_{\alpha,\beta}(\zeta; \sigma, \mu) = \exp \left[-\sigma^\alpha |k|^\alpha \left(1 - i\beta \text{sign} k \tan \frac{\pi \alpha}{2}\right) + i\mu k\right]$$
Following the Markovian character of the stochastic dynamics—at sufficiently high barriers—the time dependence of the survival probabilities within the potential well assume an exponential law.

B. Dybiec, E. G-N, P. Hänggi, PRE, 73, 046104 (2006)
B. Dybiec, E. G-N, P. Hänggi, PRE, 75, 021109 (2007)
**Langevin equation**

\[ \dot{x}(t) = -V'(x, t) + \zeta(t) \quad \Rightarrow P(x, t) \]

**Fokker-Planck equation**

\[ \frac{\partial P(x,t)}{\partial t} = \left[ \frac{\partial}{\partial x} V'(x, t) + D \frac{\partial^2}{\partial x^2} \right] P(x, t) \]

**Fractional Fokker-Planck equation**

\[ \frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x} V'(x)P(x, t) + \sigma^\alpha \frac{\partial^\alpha P(x,t)}{\partial |x|^{\alpha}} \], where

\[ \frac{\partial^\alpha}{\partial |x|^{\alpha}} f(x) = - \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ikx} |k|^{\alpha} \hat{f}(k) \]

Fine-tuning to Lévy-white noises: resonant activation and stochastic resonance
Introduction

Levy noise induced effects

Summary

Levy noises

Approximation to SODE driven by Levy noises

Integration scheme

\[ \eta(t) = \int_0^t V_0(x(s), s) \, ds + \int_0^t dL_{\alpha, \beta}(s) \]

\[ x(t) \approx \int_0^t V'(x(s), s) \, ds + \sum_{i=0}^{N-1} \Delta s^{1/\alpha} \xi_i \]

\[ \xi_i \sim S_{\alpha}(\sigma, \beta, \mu = 0) \text{ and } (N - 1) \Delta s = t. \]

B. Dybiec, E.G-N, I.M. Sokolov, PRE 78 011117 (2008)
• Systems embedded in noisy environments may enhance sensitivity
• Counterintuitively, despite their pathological character (diverging moments) Lévy fluctuations may induce better signal transmission

• Lévy white noises acting in nonlinear dynamic systems exhibit positive, ordering effects: stochastic resonance, resonant activation, synchronization and directionality of transport (ratcheting effect)
Equilibrium conditions and linear response...

\[ S_\nu(E_{\nu 1}, E_{\nu 2}, \ldots) \]

\[ S_{\text{total}} = \sum_\nu S_\nu(E_{\nu 1}, E_{\nu 2} \ldots) \]

\[ U_1 \oplus U_2 \quad \text{isolated} \]

\[ E_j = E_{1j} + E_{2j} = \text{const} \]

\[ \delta S_1(E_{1j}) + \delta S_2(E_{2j}) = 0 \quad \delta E_{1j} + \delta E_{2j} = 0 \]

\[ \Rightarrow \left( \frac{\partial S_1}{\partial E_{1j}} - \frac{\partial S_2}{\partial E_{2j}} \right) \delta E_{1j} = I_{1j} - I_{2j} \equiv 0 \quad \forall \delta E_{1j} \]
Nonequilibrium and linear response...

Lars Onsager (1903-1976)

\[ X_j \equiv \left( \frac{\partial S_1}{\partial E_{1j}} - \frac{\partial S_2}{\partial E_{2j}} \right) \delta E_{1j} = I_{1j} - I_{2j} \neq 0 \]

"thermodynamic forces"

thermodynamic forces generate fluxes

In consequence, entropy production given by a product of force and conjugated flux

\[ \frac{d E_{1,j}}{dt} = \Phi_j \]

\[ \frac{dS}{dt} = \sum_j \frac{\partial S}{\partial E_{1,j}} \frac{d E_{1,j}}{dt} = \sum_j X_j \Phi_j \]

Onsager (~1932) theory for weak \( X \) forces forsees...

\[ \Phi_j = \sum_k L_{k,j} X_k \Rightarrow \frac{dS}{dt} \approx \sum_k L_{k,j} X_k X_j \]

with the IIInd law

\[ \frac{dS}{dt} \geq 0 \]

D. Reguera, J.M. Rubi, J.M.G. Vilar

Nonequilibrium conditions and linear response? Fluctuation-dissipation theorem?

\[ f(t) = f_0 \Theta(-t) \]

\[ \langle x(t) \rangle = \int dx' \int dx \ x' p(x', t | x, 0) \tilde{p}(x, 0) \]

\[ \tilde{p}(x, 0) = \frac{e^{-\beta H(x)}}{\int dx' e^{-\beta H(x')}} = \frac{e^{-\beta[H_0(x) + xf_0]}}{\int dx' e^{-\beta H(x')}} \]

weak perturbation

\[ e^{-\beta xf_0} \approx 1 - \beta xf_0 \]

\[ \tilde{p}(x, 0) \approx p_0(x)(1 - \beta f_0 x) \]

\[ \langle x(t) \rangle = \int dx' \int dx \ x' p(x', t | x, 0) p_0(x)(1 - \beta f_0 x) = \]

\[ \langle x \rangle_0 - \beta f_0 \langle x(t)x(0) \rangle_0 \]
Linear response, fluctuation-dissipation theorem?

On the other hand...

\[ \langle x(t) \rangle = \langle x \rangle_0 + \int_{-\infty}^{t} f(\tau) \chi(t - \tau) d\tau \]

\[ f_0 \int_{0}^{\infty} d\tau \Theta(\tau - t) \chi(\tau) = \beta f_0 \langle x(t)x(0) \rangle_0 \]

\[ -\chi(t) = \beta \frac{d}{dt} \langle x(t)x(0) \rangle_0 \]

FDT relates susceptibility to correlations measured in the reference unperturbed state

\[ \langle A(t) \rangle - \langle A \rangle_0 \approx \int_0^t \chi_{A,\gamma}(t - t') \delta \lambda_\gamma(t') dt', \quad (1) \]

\[ \chi_{A,\gamma}(t - t') = \frac{d}{dt} \langle A(t)X_\gamma(t') \rangle_0, \quad (2) \]

\[ X_\gamma(x) = -\frac{\partial \ln \rho_{ss}(x; \bar{\lambda})}{\partial \lambda_\gamma} \bigg|_{\bar{\lambda} = \bar{\lambda}_0} = \frac{\partial \phi}{\partial \lambda_\gamma}. \quad (3) \]

\[ \phi \equiv -\ln \rho_{ss} \]

\[ \rho_{ss}(x; \bar{\lambda}) = \frac{\exp[-\beta \mathcal{H}(x; \bar{\lambda})]}{Z(\beta, \bar{\lambda})} \]

\[ X_\gamma(x) = \frac{1}{kT} \left[ \frac{\partial \mathcal{H}(x; \bar{\lambda}_0)}{\partial \lambda_\gamma} - \left\langle \frac{\partial \mathcal{H}(x; \bar{\lambda}_0)}{\partial \lambda_\gamma} \right\rangle_0 \right] \]

FDT: measurable macroscopic physical quantities related to correlations functions of spontaneous fluctuations

\[
X_\gamma(x) = \frac{1}{kT} \left[ \frac{\partial \mathcal{H}(x; \tilde{\lambda}_0)}{\partial \lambda_\gamma} - \left\langle \frac{\partial \mathcal{H}(x; \tilde{\lambda}_0)}{\partial \lambda_\gamma} \right\rangle_0 \right]
\]

\[
X_\gamma(x) = \frac{1}{kT} \left. \partial \left[ \mathcal{H}(x; \tilde{\lambda}) - F(\beta, \tilde{\lambda}) \right] \right|_{\tilde{\lambda}=\tilde{\lambda}_0}
\]

\[
\rho_{ss} = \exp[-\beta \mathcal{H}] Z^{-1} \quad T(t) \to T + \delta T
\]

\[
\delta \mathcal{H} = \alpha(t) \delta T
\]

Isochoric specific heat estimated by analysing fluctuations in the steady state

\[
\alpha(t) = \frac{1}{kT^2} \left\langle \delta \mathcal{H}(0) \delta \mathcal{H}(0) \right\rangle_0
\]
Conjugate variable...

\[
\begin{aligned}
\dot{x}(t) &= -ax + f(t) + \zeta(t) \\
 x(0) &= x_0
\end{aligned}
\]

\[\langle X(t) \rangle = \int_{-\infty}^{\infty} X(x)p(x, t) \, dx\]

\[\hat{p}(k, t) = \exp \left[ ik\mu(t) - \sigma^\alpha(t)|k|^\alpha \left( 1 - i\beta \text{sign}(k) \tan \frac{\pi\alpha}{2} \right) \right]\]

\[p_{1,0}(x, t|x_0, 0) = \frac{\sigma(t)}{\pi} \frac{1}{[x - \mu(t)]^2 + \sigma^2(t)}\]

\[X_C = -\frac{2x}{a[x^2 + (\sigma_0/a)^2]}\]

\[\chi(t) = \frac{d}{dt} \langle X(t)X(0) \rangle_0 = -\frac{a}{2\sigma_0^2} e^{-at}\]

\[\langle X(t) \rangle_{LR} = \int_0^t \chi(t-s)f(s) \, ds\]
Response of conjugate variable to external drivings:
solid lines -- exact result
dotted lines -- result constructed by use of the linear response theory

\[ \langle X \rangle = \frac{\sigma_0}{a \pi} \int_{-\infty}^{\infty} \frac{dx}{[x - f/a]^2 + (\sigma_0/a)^2} \frac{2x}{a^2 + (\sigma_0/a)^2} \]

Conjugate variables reflect change in the PDF under the perturbation

Despite the system is plagued by divergent moments, the generalized FDT properly captures dynamical response

Drawbacks: interpretation of \( \langle X \rangle \)
Two independent noises: Cauchy and Gauss

\[ \dot{x}(t) = \mu_0 - ax + f(t) + \xi_c(t) + \xi_g(t) \]

\[ \hat{p}(k, t) = e^{ik\mu(t) - \sigma^2(t)|k|^2 - \gamma(t)|k|} \]

for \( f(t) = 0 \)

\[ p_s(x) = \frac{1}{2\sqrt{\pi}\sigma_\infty} \text{Re} w\left( \frac{-x + i\gamma_\infty}{2\sigma_\infty} \right) \]

\[ w(x) := e^{-x^2} \text{erfc}(-ix) \]
Two independent noises: Gaussian and Cauchy

\[
X_{gc} = -\frac{x}{2\sigma_\infty^2 a} - \frac{\gamma_\infty}{2\sigma_\infty^2 a} \operatorname{Im} w\left(\frac{-x+i\gamma_\infty}{2\sigma_\infty}\right)
- \frac{\gamma_\infty}{2\sigma_\infty^2 a} \operatorname{Re} w\left(\frac{-x+i\gamma_\infty}{2\sigma_\infty}\right)
\]

\[
\lim_{\gamma_0 \to 0} X_{gc} = -\frac{x}{2a\sigma_\infty^2} = -\frac{x}{\sigma_0^2}
\]

\[
\lim_{\sigma_0 \to 0} X_{gc} = -\frac{2ax}{\gamma_0^2 + a^2 x^2}
\]

Conjugate variable...
\[ f(t) = \sin(t)/10 + t/10 \]
• Nonequilibrium steady states are fascinating systems to study...
• The generalized FDT can be applied to (linear) systems driven by Lévy noises
• The conjugate variables represent change in PDF under perturbation (in equilibrium related to energy absorbed from perturbations)
• Interpretation of conjugate variables is not straightforward...
Special thanks:

B. Dybiec, W. Ebeling, J. Parrondo, Ł. Kuśmierz
Breakdown of common thermodynamics?

\[ dU = \delta Q + \delta W \]

\[ \frac{\delta Q}{T} = dS \quad \Delta S \geq 0 \]

In far-from equilibrium situations a common definition of temperature does not make sense.

Traditional thermodynamics does not describe transitions between metastable states.

Theory is not suitable for description of ordering phenomena in nonequilibrium states.

DISSIPATION RELATED TO ORDER!

- Axiomatic formulation: „traditional” thermodynamics is an elegant mathematical theory
- Key words: a state, a state-function, equation of state (characteristic of state quantities)
- Infinitesimal, adiabatic changes of state are time-reversible

J.M. Rubi, Scientific American, 2008
Importance of anomalous transport...

I. Bronstein, Y. Israel et al. PRL, 103, 018102 (2009)