CENTRAL LIMIT THEOREM

1. A coin is tossed 400 times. Use the normal-curve approximation to the Bernoulli scheme to find the probability of obtaining
(a) between 185 and 210 heads inclusive *hint: consider an interval: (184.5,210.5)*;
(b) exactly 205 heads *hint: 205 $\equiv$ an interval: [204.5,205.5)*;
(c) less than 176 heads (i.e., ≤ 175.5).

   Hint: The normal-curve approximation to the Bernoulli scheme means, that we treat the distribution as a Gaussian with $\mu = np = 200$ and $\sigma^2 = npq = 100$. Then we follow the schemes as in the set no.6

   answers: (a) 0.7925; (b)0.0352; (c) 0.0071

2. An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed, with mean μ (expected value) equal to 800 hours and a standard deviation of 40 hours. Find the probability that a random sample of 16 bulbs will have an **average** life of less than 775 hours.
Hint: The average life, $\bar{X}$ will be approximately normal with $\mu_{\bar{X}} = 800$ and $\sigma_{\bar{X}} = 40/\sqrt{16} = 10$. The answer: P = 0.0062

3. A manufacturing process produces cylindrical component parts for the automotive industry. It is important that the process produce parts having a mean of 5 millimeters. The engineer involved conjectures that the population mean (i.e., the expected value for the whole production) is 5.0 mm. An experiment is conducted in which 100 parts produced by the process are selected randomly and the diameter measured on each. It is known that the population standard deviation (i.e. the standard deviation characteristic for every individual part) σ is 0.1. The experiment gives a sample average diameter $\bar{x} = 5.027$ mm. Does this sample information appear to support or refute the engineer's conjecture?
Hint: In other words, how likely is it that one can obtain $\bar{x} \geq 5.027$ for the sample size $n = 100$ if the population mean $\mu = 5.0$?

   we standardise:
$$Z = \frac{5.027 - 5.00}{\sigma/\sqrt{n}} = \frac{5.027 - 5.00}{0.1/\sqrt{100}} = 2.7$$
$$P = P(Z > 2.7) = 1 - P(Z < 2.7) = 1 - 0.9965 = 0.0035.$$

   Answer: The calculated P is 0.0035 – it cannot be treated as a supporting evidence to the conjecture that the $\mu = 5.00$.

4. Suppose a machine set for filling concrete-mix bags yields a weight W with $E(W) = 16$ kilograms and $\sigma(W) = 0.2$ kilogram. A container of mix contains 48 bags. (1) Describe the distribution of the total mass M of the container.

   answer: (approximately) normal, expected value $\mu = 48 \times E(W) = 768$ kilograms. Variance: $\sigma^2(M) = 48 \times \sigma^2(W) = 48 \times 0.04 = 1.92$, hence $\sigma(M) = 1.39$

   (2) Compute the probability that the total weight of container exceeds 771.2 kilograms; we standardise:
$$Z = \frac{771.2 - 768}{\sigma(M)} = \frac{771.2 - 768}{1.39} = 2.3$$
$$P = P(Z > 2.3) = 1 - P(Z < 2.3) = 1 - 0.9893 = 0.0107.$$

5. The (assumed) mean diameter μ of marbles manufactured at a particular toy factory is 0.850 cm with a standard deviation $\sigma = 0.01$ cm. What is the probability of selecting a random sample of $n = 100$ marbles that has the mean diameter greater than 0.851 cm?

   standardization: $Z = \dfrac{0.851 - \mu}{\sigma/\sqrt{n}} = \ldots 1$; probability $Z > 1$ from the normal distribution tables is 0.159

6. A research worker wishes to estimate the mean of a population using a sample large enough that the probability will be 0.95 that the sample mean will not differ from the population mean by more than 25% of the standard deviation. How large a sample should he take?

   Solution:

   We have $\mathcal{P}(|\bar{X}-\mu| < \sigma/4) = 0.95$; divide by $\sigma/\sqrt{n}$

   $\mathcal{P}\left(\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \dfrac{\sqrt{n}}{4}\right) = 0.95$

   But $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ has a standard normal distribution, i.e. The question is:

   $\mathcal{P}(-1.96 <$ Standard Normal $< 1.96) = 0.95$ hence $1.96 = \sqrt{n}/4$; $n = 62$

7. Two random samples of size 100 are drawn from two populations P_1 and P_2 and their means $\bar{X}_1$ and $\bar{X}_2$, computed. If $\mu_1 = 10,\ \sigma_1 = 2,\ \ \mu_2 = 8,\ \sigma_2 = 1$, find:

   (a) $E(\bar{X}_1 - \bar{X}_2)$;

   answer $E(\bar{X}_1 - \bar{X}_2) = \mu_1 - \mu_2$

   (b) $\sigma_{\bar{X}_1-\bar{X}_2}$;

   answer $\sigma^2_{\bar{X}_1-\bar{X}_2} = \sigma_1^2/n_1 + \sigma_2^2/n_2 = \ldots = \dfrac{\sqrt{5}}{10}$;

   (c) the probability that the difference between a given pair of sample means is less than 1.5, i.e. $\mathcal{P}(-1.5 < \bar{X}_1 - \bar{X}_2 < 1.5)$;

   we have: $\mathcal{P}(-1.5 < \bar{X}_1 - \bar{X}_2 < 1.5 = ?$; dividing all three members by $\sigma_{\bar{X}_1-\bar{X}_2}$ and subtracting from all three denominators $E(\bar{X}_1 - \bar{X}_2) = \mu_1 - \mu_2$ we have

$$\mathcal{P}\left(\frac{-1.5-2}{\sqrt{5}/10} < \frac{(\bar{X}_1-\bar{X}_2) - E(\bar{X}_1-\bar{X}_2)}{\sigma^2_{\bar{X}_1-\bar{X}_2}} < \frac{1.5-2}{\sqrt{5}/10}\right)$$

$$= \mathcal{P}\left(-15.652 < \frac{(\bar{X}_1-\bar{X}_2) - E(\bar{X}_1-\bar{X}_2)}{\sigma^2_{\bar{X}_1-\bar{X}_2}} < -2.236\right) \approx \mathcal{P}\left(\frac{(\bar{X}_1-\bar{X}_2) - E(\bar{X}_1-\bar{X}_2)}{\sigma^2_{\bar{X}_1-\bar{X}_2}} < -2.236\right)$$

   from tables of normal distirbution $= \mathcal{P} = 0.0127$.

   answer: employ the technique as in c). (d) the probability that the difference between a given pair of sample means is greater than 1.75 but less than 2.5, i.e. $\mathcal{P}(-1.75 < \bar{X}_1 - \bar{X}_2 < 2.5)$.