SOME MOST POPULAR DISTRIBUTIONS OF RANDOM VARIABLES

1.ONE-POINT (single-valued) RV: $\mathcal{P}(X = x_0) = 1$

$$F(x) = \begin{cases} 0 & x \leq x_0 \\ 1 & x > x_0 \end{cases}$$

$E\{X\} = x_0$; VAR(X) = 0.

2.TWO-POINT (two-valued):

$$\mathcal{P}(X = x_1) = p, \quad \mathcal{P}(X = x_2) = q = 1 - p$$

let's put: $x_1 = 1$, $x_2 = 0$

$$E\{X\} = p, \quad VAR(X) = p(1-p), \quad \mu_3 = p(1-p)(1-2p)$$

BERNOULLI (BINOMIAL) DISTRIBUTION

$$E = A + \bar{A}; \quad \mathcal{P}(A) = p; \quad \mathcal{P}(\bar{A}) = q = 1 - p$$

RV $X = \sum_i X_i$ — a Bernoulli sequence of trials:

$$X_i = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } \bar{A} \end{cases}$$

$$W_k^n \equiv \mathcal{P}(X = k) = \binom{n}{k} p^k q^{n-k}$$

$$E\{X_i\} = p \cdot 1 + q \cdot 0 = p; \quad VAR(X_i) = \ldots = pq$$

$$E\{X\} = np \quad \sigma^2(X) = npq$$

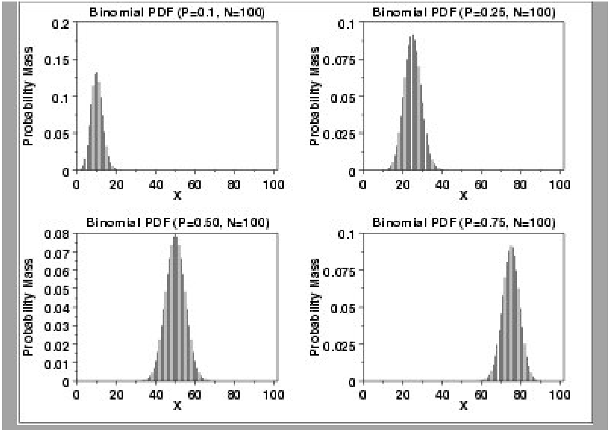
(the variables X_i are independent of each other!)

BERNOULLI, cntd.

A Bernoulli trial process is a sequence of independent and identically distributed RVs: $X_1, \ldots, X_n$ — n repetitions of an experiment, under identical conditions, with each experiment producing only two outcomes: success ($\mathcal{P} = p$) or failure ($\mathcal{P} = q = 1 - p$).

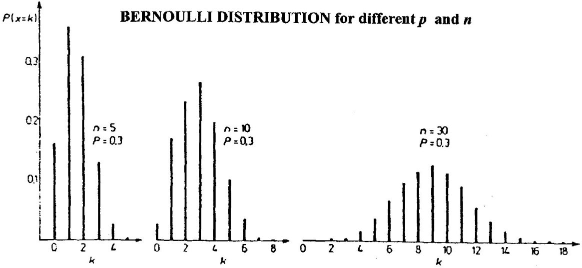
The binomial (Bernoulli) distribution...

here come few graphs of binomial (Bernoulli) distribution for different p and n values:



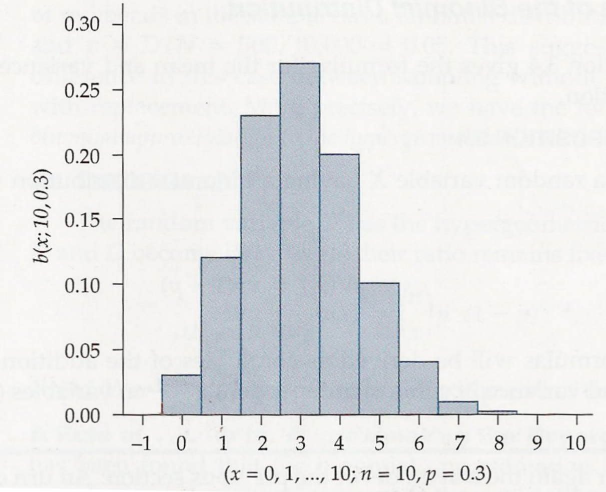
http://www.itl.nist.gov; Jan 5th 2012

more graphs of binomial (Bernoulli) distribution for different p and n values:

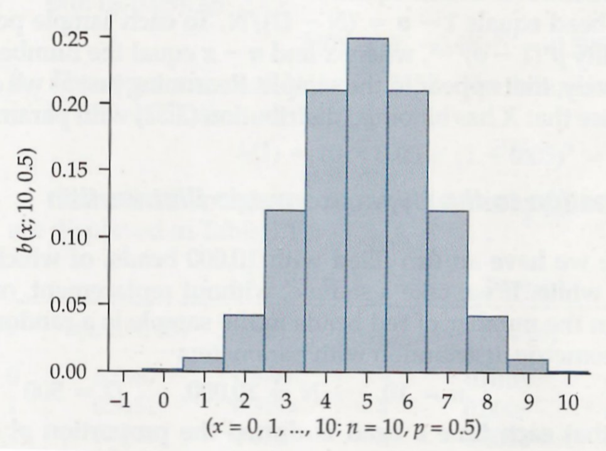


The binomial (Bernoulli) distribution. . .

more graphs of binomial (Bernoulli) distribution for different p and n values: (W.A. Rosenkrantz, Introduction to Probability and Statistics, Mc-Graw-Hill, 1997)

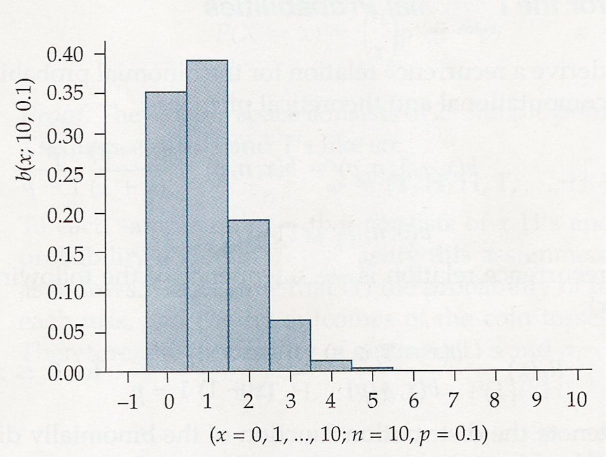


more graphs of binomial (Bernoulli) distribution for different p and n values: (W.A. Rosenkrantz, Introduction to Probability and Statistics, Mc-Graw-Hill, 1997)



The binomial (Bernoulli) distribution...

one more graph of binomial (Bernoulli) distribution for different p and n values: (W.A. Rosenkrantz, Introduction to Probability and Statistics, Mc-Graw-Hill, 1997)



$$E = A_1 + A_2 + \ldots + A_N; \quad \mathcal{P}(A_k) = p_k; \quad \sum_{k=1}^{N} p_k = 1$$

$$\underbrace{p_1 \cdot p_1 \ldots p_1}_{k_1} \cdot \underbrace{p_2 \cdot p_2 \ldots p_2}_{k_2} \cdot \underbrace{p_3 \cdot p_3 \ldots p_3}_{k_3} \cdots \underbrace{p_N \cdot p_N \ldots p_N}_{k_N}$$

we have

$$k_1 + k_2 + k_3 + \ldots + k_N = n$$

so

$$W^n_{k_1,k_2,\ldots,k_n} = \frac{n!}{\prod_{j=1}^{N} k_j!} \prod_{j=1}^{N} p_j^{k_j}$$

$$X_{ij} = \begin{cases} 1 \text{ the outcome of i-th trial} = A_j \\ 0 \text{ otherwise} \end{cases}$$

$$X_j = \sum_{i=1}^{n} X_{ij}$$

the expected value:

$$E\{X_j\} = \hat{x}_j = n \cdot p_j$$

MULTINOMIAL DISTRIBUTION, cntd.

covariances and variances:

$$c_{ij} = np_i(\delta_{ij} - p_j)$$

— this means that any two events $\{A_i, A_j\}$ cannot be independent (unless $p_i \equiv 0$ or $p_j \equiv 0$.)

p_j — probability of the event A_j can be associated with the frequency $\nu_j \equiv \nu$:

$$\nu = \frac{1}{n}\sum_{i=1}^{n} X_{ij} = \frac{1}{n}X_j$$

$$E\{\nu\} = \frac{1}{n}E\{X_j\} = p_j$$

what about the error?

$$\sigma^2(\nu) = \sigma^2\left(\frac{X_j}{n}\right) = \frac{1}{n^2}\sigma^2(X_j) = \frac{1}{n^2}np_j(1-p_j) = \frac{1}{n}p_j(1-p_j)$$

$$\sigma(\nu) \propto \frac{1}{\sqrt{n}}$$

this is one of the conclusions which may be drawn from the so–called LAW OF BIG NUMBERS: the error acompanying an estimation based on an ensemble of size n is $\propto 1/\sqrt{n}$.

Imagine a bag with R red balls and $N - R$ black balls (total – N balls). We select at random a sample of size n.
If we denote by x the number of red balls we have:
$\max(0, n - (N - R)) \leq x \leq \min(n, R)$ also

$$h(x) = \frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}}.$$

The expected value and variance are

$$E(X) = n \times \frac{R}{N} \qquad VAR(X) = \frac{N-n}{N-1} \times \frac{R}{N}\left(1 - \frac{R}{N}\right).$$

When the sample size (n) is less than 5 percent of the population size (N) the Hypergeometric Distribution is quite well approximated by the binomial (Bernoulli) distribution with $p = R/N$.

Computer mice are packed in lots of 100. Ten mice are selected and tested. Any one (or more) is found to be ill-functioning the whole lot is rejected, i.e. the lot may be accepted only if none of the tested mice is defective.
There are 6 defective mice in the lot. What is the probability of accepting the whole lot?

$$\mathcal{P}(\text{accepting}) = \mathcal{P}(X = 0) = \frac{\binom{6}{0}\binom{94}{10}}{\binom{100}{10}}.$$

$\mathcal{P} \approx 0.51$; what if we increase the sample size $n = 20$?

$$\frac{\binom{6}{0}\binom{94}{20}}{\binom{100}{20}}$$

roughly 25 percent...