OTHER DISTRIBUTIONS OF RANDOM VARIABLES

The normal distribution. . .

is by far the most important of all distribution of RV. For the continuous RV we have, however, several other distributions that are also of importance. These are, among others:

- Student's distribution
- Chi-square distribution
- Snedecor ($\mathcal{F}$) distribution

All these distributions will be discussed later on, when we shall to come to deal with them in connection with some specific statistical problems. Here, we should characterise briefly other pdf functions of the continuous RV. These will be:

The **exponential distribution**. . .

. . . with the density function

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $\beta > 0$. By the usual formulae it is very easy to show that $E(X) = \beta$ and $VAR(X) = \beta^2$.

There is a direct connection between the exponential distribution and the Poisson distribution (process). We may remember that the unique parameter of Poisson distribution λ can be interpreted as a mean number of events per unit time.
So, if we consider a time interval of the length t the number of events should be $N = \lambda \cdot t$.
Consider now a RV described by the time required for the first event to occur, X. The probability that the length of time until the event will exceed x is equal to the probability that no Poisson events will occur in x.

The latter is:

$$\mathcal{P}(0; \lambda t) = e^{-\lambda t} \frac{(\lambda t)^0}{0!} = e^{-\lambda t}$$

and we have: $\mathcal{P}(X \geq x) = e^{-\lambda t}$ and the cumulative distribution function for X is

$$F(x) = \mathcal{P}(0 < X < x) = 1 - e^{-\lambda t}.$$

It is sufficient to differentiate the above formula to obtain the df for X:

$$f(x) = \lambda e^{-\lambda t}$$

which is the exponential distribution with $\beta = 1/\lambda$.

In a space shuttle a critical component of an experiment has an expected lifetime $\lambda = E(T) = 10$ days and it is exponentially distributed with $\beta = 1/\lambda = 0.1$. The mission is scheduled for 10 days. How many $(n - 1 \ = ?)$ spare parts should we put on the board if we want to have the probability of success at least equal to 0.99? (All together we will be using n components).
notation: T_i – the lifetime of the i-th component; W_n – the total lifetime for the n-th component:

$$W_n = T_1 + T_2 + \ldots + T_n.$$

Both T_i and W_n have the same probability density function.
$X(t)$ – number of events (break-downs) occuring in time t.

$$\mathcal{P}(W_n > t) = \mathcal{P}(X(t) < n) = \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

right-hand member: possible numbers of events (failures) during the time t.

– example from W. Rosenkrantz book, cntd.

$$\mathcal{P}(W_n > t) = \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!} = \dots .$$

$$\mathcal{P}(W_n > t) = \boxed{\begin{array}{c} \lambda = 0.1 \\ t = 10 \end{array}} = e^{-1}\left[1 + 1 + \dots + \frac{1}{(n-1)!}\right].$$

If we use two spares ($n - 1 = 2$; $n = 3$) the probability ia a meagre 92%!!
To have the probability of at least 99 percent we must put $n = 5$.
And this means four (!) spares (plus the component in the apparatus).

The exponential distribution...

...is a special case of the so-called *gamma distribution*. The latter is directly connected with the definition of the Euler gamma function:

$$\Gamma(\alpha) \stackrel{\text{def}}{=} \int_0^\infty e^{-t} t^{\alpha-1} dt$$

Gamma function in turn can be viewed as a generalisation of the factorial "function" for non-integer numbers. We have:

$$\boxed{\Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$$

$\Gamma(n+1) = n!; \quad \Gamma(1/2) = \sqrt{\pi}.$

From the above definition and the normalisation condition we see that the properly normalised gamma density distribution will be given by:

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $\beta > 0$ and $\alpha > 0$. Again, using the usual formulae it is very easy to show that $E(X) = \alpha\beta$ and $VAR(X) = \alpha\beta^2$.

The exponential df is a special case for $\alpha = 1$.

NB. The so-called *Weibull distribution* is also a generalisation of both exponential and gamma distributions. Its density function is given by

$$f(x) = \begin{cases} \beta\alpha x^{\beta-1}e^{-\alpha x^{\beta}} & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Continuous uniform distribution

is a quite important distributions that describes the case when the probability of finding X in an interval of the length $L = b - a$ is constant for all the X's values: $a \leq x \leq b$. We have

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0, & \text{elsewhere} \end{cases}$$

The expected value results from symmetry: $E(X) = (b-a)/2$ and the variance is $VAR(X) = (b-a)^2/12 = L^2/12$.
The latter result can be deduced from . . . physical formulae. (Consider the value of the moment of inertia of a homogeneous rod, which rotates around a perpendicular axis that cuts the rod into two halves.)
The standard deviation consequently is

$$\sigma(X) = \sqrt{VAR(X)} = \frac{L}{2\sqrt{3}}.$$