Towards the Heider balance – asymmetric social relations

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*in cooperation with*

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the Heider (structural) balance:

- world divided into two groups, internally friendly and mutually hostile

The looking-glass self:

- I am not who I think I am
- I am not who you think I am
- I am who I think you think I am
Networks of social relations
How to reach the balance:
  - why?
  - from triads to network
  - an example
  - discrete and continuous dynamics
  - jammed states
  - asymmetric relations
Self-evaluation
Some stable configurations and an algorithm to classify them
More configurations and why they are so similar?
A tentative application
1928-1933: first sociograms
- Sing Sing prison,
- a reformatory for delinquent girls,
- public and private Brooklyn schools.

https://archive.org/stream/whoshallsurviven00jlim#page/38/mode/2up
www.slate.com/articles/technology/future_tense/2014/10/j_l_moreno_a_psychologist_s_30s_experiments_invented_social_networking.html
Networks of social relations: an example

Zachary karate club – first 10 rows/columns

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0 1 1 1 1 1 1 1 1 0
1 0 1 1 0 0 0 1 0 0
1 1 0 1 0 0 0 1 1 1
1 1 1 0 0 0 0 1 0 0
1 0 0 0 0 0 1 0 0 0
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1 0 1 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0
```

1 means that „the two individuals being represented consistently interacted in contexts outside those of karate classes, workouts, and club meetings.”


[vlado.fmf.uni-lj.si/pub/networks/data/ucinet/zachary.dat]

„Students, teachers and administrators of a college karate club that suffered a strong conflict that led to the club excision in two groups. Node colors in the graph describe the final faction of each individual. „

[webpage of José Javier Ramasco, ifisc.uib-csic.es/~jramasco/]
Towards structural balance – why?

Cognitive dissonance about friends and enemies is removed if:

- a friend of my friend is my friend,
- a friend of my enemy is my enemy,
- an enemy of my friend is my enemy,
- an enemy of my enemy is my friend.
Towards structural balance

**Cartwright – Harary Theorem:** If a labeled complete graph is balanced, then either all relations are friendly, or else the nodes can be divided into two groups, $X$ and $Y$, such that every pair of people in $X$ like each other, every pair of people in $Y$ like each other, and everyone in $X$ is the enemy of everyone in $Y$.

Three Emperor’s League 1872–81

Triple Alliance 1882

German-Russian League 1890

French-Russian Alliance 1891–94

Entente Cordiale 1904

British Russian Alliance 1907

Towards structural balance: discrete MC algorithms

A. Local Triad Dynamics

B. Constrained Triad Dynamics $\equiv$ Metropolis algorithm, where

1. Find an unbalanced triad
2. Change a random link
3. If $U$ increases, withdraw the change
4. If $U$ is not changed, withdraw the change with $p=1/2$

$$U = -\sum_{k=1}^{N} x(ik)x(kj)x(ji)$$

Discrete MC algorithms: jammed states

In the Constrained Triad Dynamics, some configurations remain unbalanced and stable.

FIG. 7: Examples of jammed configurations for $N = 9$ (only friendly links are displayed). (a) A jammed configuration that appeared in simulations. (b) A jammed state consisting of three mutually antagonistic cliques. (c) A jammed state derived from (b) in which the top clique from (b) is friendly toward the remaining two cliques.

Towards structural balance: continuous dynamics

\[
\frac{dx_{ij}}{dt} = \sum_{k \neq i, \neq j}^{N-2} x_{ik} x_{kj} = -\frac{\partial U}{\partial x_{ij}}
\]

or just the matrix Riccati equation

\[
\frac{dX}{dt} = X^2
\]

with the solution

\[
X(t) = X(0)[I - X(0)t]^{-1}
\]

Towards structural balance: continuous dynamics

\[ \frac{dx_{ij}}{dt} = (1 - x_{ij}^2) \sum_{k \neq i, \neq j} x_{ik} x_{kj} \]

with the asymptotic solutions \( x_{ij} = \pm 1 \)

such that \( x_{ij} = \text{sign} \left( \sum_{k \neq i, \neq j} x_{ik} x_{kj} \right) \)

which appears to be balanced in most cases.

Continuous dynamics: jammed states

The condition \( x_{ij} = \pm 1 = \text{sign} \sum_{k \neq i, \neq j}^{N-2} x_{ik} x_{kj} \) provides the stability.

\[
A_{ij} = \sum_{k \neq i, \neq j}^{N-2} x_{ik} x_{kj}
\]

Example 1:

\[
A(\text{---}) = +4
\]

\[
A(\text{---}) = -4
\]

Example 2:

\[
A(\text{---}) = +7
\]

\[
A(\text{---}) = -1
\]
unreciprocated relations: an example

Sampson data - first 10 rows/columns

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Who likes whom? Each member ranked only his top three choices on that tie.

the same dynamics \[ \frac{dx_{ij}}{dt} = (1 - x_{ij}^2) \sum_{k \neq i, j}^{N-2} x_{ik} x_{kj} \]

for asymmetric relations \( x_{ij} \neq x_{ji} \)

lead to \( x_{ik} = \pm 1 \) in the stationary states, with the same stability conditions
A SELF-EVALUATION INDEX

where only „significant others” are counted

\[ y_i = \frac{1}{2} \sum_{k \neq i}^{N-1} (1 + x_{ik}) x_{ki} \]

\[ y_i \geq 0 \rightarrow \text{self-evaluation positive or zero} \]
\[ y_i < 0 \rightarrow \text{self-evaluation negative} \]
\[ (i \text{ frustrated}) \]
some stationary states

\[ y = 1 - N_2 \]

\[ y = 1 - N_1 \]

\[ y = -N_1 \]
For higher $N$, we meet the problem of graph classification.

An algorithm for given $N$:

a) Set a random asymmetric initial state $\{x_{ij}(t=0)\}$
b) Find the stationary solution of the dynamics $\{x_{ij}(t=\infty)\}$
c) Classify nodes according to their number of positive links
d) Do the neighbors of nodes in the same class belong to the same classes?
   If NOT, refine the classification. Continue until YES.
e) The graph is classified as the list of classes of nodes, with the numbers of nodes in each class, and the relations between the classes.

[M. J. Krawczyk, Physica A 390 (2011) 2181]
class equivalence

\[ y = -N_1 \]
more stationary states

balanced, symmetric:

unbalanced, asymmetric:

CII: $N_1 > N_2 + 2$

CIII: $N_1 + N_2 > N_3 + 2$

CIV: $N_1 + N_2 > N_3 + N_4 + 2$
The self-evaluation index

balanced, symmetric

unbalanced, asymmetric

\[ y = -N_1 \]

\[ y = N_4 - N_2 \]
The frequencies of $\text{CIII}$ and $\text{CIV}$ for $N=7$ and $N=9$. 
On the frequency of the stationary states

balanced, symmetric

unbalanced, asymmetric
The frequencies of CIV for $N=41$, 55 and 71.
The number of ways of partitioning a set of $n$ distinguishable elements into $k$ nonempty sets

**Stirling numbers of 2-nd kind $S(n,k)$**

\[
S(n, 2) = 2^{n-1} - 1
\]

\[
S(n, 3) = \frac{1}{6} (3^n - 3 \cdot 2^n + 3)
\]

\[
S(n, k) = S(n-1, k-1) + kS(n-1, k)
\]

$S(10, 3) = 9330$

$S(10, 4) = 34105$

$S(20, 3) = 580606446$

$S(20, 4) = 45232115901$

$S(30, 3) = 34314651811530$

$S(30, 4) = 48004081105038305$

the Sampson data: \(18 \times 18\) non-symmetric, valued rankings

SAMPLK/SAMPDLK : whom you like/dislike most? \(3,2,1\)
SAMPES/SAMPDES : whom you estimate most/least? \(3,2,1\)
SAMPIN/SAMPNIN : positive/negative influence \(3,2,1\)
SAMPPR/SAMPNPR : whom you praise/blame most? \(3,2,1\)

vlado.fmf.uni-lj.si/pub/networks/data/ucinet/ucidata.htm#sampson

matrices too thin: if we remove the 1-st monk, then:

\[
\frac{(SAMPIN - SAMPNIN)}{5} \rightarrow CIII \\
\frac{(SAMPLK - SAMPDLK)}{5} \rightarrow CIV \\
\frac{(SAMPES - SAMPDES)}{5} \rightarrow CIV \\
\frac{(SAMPPR - SAMPNPR)}{5} \rightarrow CIV
\]
Following the „looking-glass self” theory, we infer on low self-evaluation from asymmetric interpersonal relations.

The process of removal of cognitive dissonance drives interpersonal relations to stationary configurations. Some of these configurations appear to be frustrating for selected individuals.

Our results allow to classify these configurations and – in principle – to identify them in social groups.

more details in arXiv:1903.12464
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