TWO–DIMENSIONAL RANDOM VARIABLE
A PAIR OF RVs: $X$, $Y$

CUMULATIVE DISTRIBUTION FUNCTION:

\[
F(x, y) = \mathcal{P}(X \leq x, Y \leq y) = \begin{cases} 
\sum_{x_i \leq x, y_k \leq y} \mathcal{P}(X = x_i, Y = y_k) & \text{(discrete RV)} \\
\int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y) \, dx \, dy & \text{(continuous RV)}
\end{cases}
\]

for a RV of discrete type we may define:

\[p_{ik} = \mathcal{P}(X = x_i, Y = y_k)\]

and for a RV of continuous type we have:

\[f(x, y) = \frac{\partial^{2} F}{\partial x \partial y}; \quad \mathcal{P}(X \in [x, x+dx] \cap Y \in [y, y+dy]) = f(x, y) \, dx \, dy\]
Marginal Distribution Functions:

\[ P(a \leq x \leq b; \ y \ \text{any value}) = P(a \leq x \leq b; -\infty \leq y \leq \infty) = \int_{a}^{b} \left[ \int_{-\infty}^{\infty} f(x, y) \, dy \right] \, dx \equiv \int_{a}^{b} g(x) \, dx \]

\[ g(x) = \int_{-\infty}^{\infty} f(x, y) \, dy \]

in an exactly analogous way:

\[ h(y) = \int_{-\infty}^{\infty} f(x, y) \, dx \]

we call \( g(x) \), \( h(y) \) marginal distribution functions of \( x \) and \( y \), respectively. Their role is exactly the same as the role of the pdf of a single RV.
conditional distributions:

\[ f(y|x_0) = \frac{f(x_0, y)}{\int_{-\infty}^{\infty} f(x_0, y) \, dy} = \frac{f(x_0, y)}{g(x_0)} \]

\[ f(x|y_0) = \frac{f(x, y_0)}{\int_{-\infty}^{\infty} f(x, y_0) \, dx} = \frac{f(x, y_0)}{h(y_0)} \]

Note: the * equalities follow from the normalisation condition, i.e.:

\[ \int_{-\infty}^{\infty} f(y|x_0) \, dy = 1 = \int_{-\infty}^{\infty} f(x|y_0) \, dx \]

Now, let's put simply \( y_0 = y \) and \( x_0 = x \). We have:

\[ h(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \int_{-\infty}^{\infty} f(y|x)g(x) \, dx \]

\[ g(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{-\infty}^{\infty} f(x|y)h(y) \, dy \]

Now if RVs X and Y are INDEPENDENT we have:

\[ f(y|x) = h(y) \quad f(x|y) = g(x) \]
conditional distributions:

On the other hand we have:

\[ f(y|x) = \frac{f(x,y)}{g(x)} = h(y) \]

so

\[ f(x,y) = g(x) \cdot h(y) \]

for the independent RVs X and Y the joint probability (density) function is a product of two corresponding marginal (density) distributions!

Note: for a discrete RV we have also marginal distributions. Let \( p_{ik} = P(X = x_i; Y = y_k); \ \sum_{i,k} p_{ik} = 1 \). Then the marginal probability for X, \( p_{i.} \) and Y, \( p_{.k} \) will be defined, respectively, as:

\[
\begin{align*}
  p_{i.} &= P(X = x_i; Y = \text{any value}) \\
  p_{.k} &= P(Y = y_k; X = \text{any value})
\end{align*}
\]

Of course, we have:

\[
\sum_i p_{i.} = \sum_k p_{.k} = 1.
\]
THE PARAMETERS OF A 2D RV \((X,Y)\)

(case of a continuous variable):

\[
E\{H(X,Y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x,y) f(x,y) \, dx \, dy
\]

The moments:

\[
\lambda_{lm} = E\{X^l Y^m\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^l y^m f(x,y) \, dx \, dy
\]

\[
\alpha_{lm} = E\{(X-a)^l (Y-b)^m\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-a)^l (y-b)^m f(x,y) \, dx \, dy
\]
THE PARAMETERS OF A 2D RV (X,Y), cntd.

the expected (mean) value of RV X:

\[ E\{X\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) \, dx \, dy = \lambda_{10} = \int_{-\infty}^{\infty} x g(x) \, dx \]

\[ E\{Y\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x, y) \, dx \, dy = \lambda_{01} = \int_{-\infty}^{\infty} y h(y) \, dy \]

central moments:

\[ a = \lambda_{10} \quad b = \lambda_{01} \]

\[ \mu_{lm} = E \{ (X - \lambda_{10})^l (Y - \lambda_{01})^m \} \]

\[ \mu_{00} = 1; \quad \mu_{10} = \mu_{01} = 0 \]

\[ \mu_{11} = COV(X, Y) \]

\[ \mu_{20} = VAR(X) \]

\[ \mu_{02} = VAR(Y) \]
THE PARAMETERS OF A 2D RV \((X,Y)\)
THE COVARIANCE AND CORRELATION OF A 2D RV:

\[
\text{COV}(X,Y) = E\{(X - \mu_X)(Y - \mu_Y)\} = \ldots = E\{XY\} - \mu_X \mu_Y
\]

note (and remember): \(\text{COV}(X, X) = \text{VAR}(X)\)

CORRELATION

\[
\rho(X,Y) \equiv \text{CORR}(X,Y) = \frac{\text{COV}(X,Y)}{\sigma(X)\sigma(Y)}
\]

it’s very easy to show that

\[-1 \leq \rho \leq +1\]

FOR 2 INDEPENDENT RVs:

\[
f(x,y) = g(x)h(y)
\]

\[
\text{COV}(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \hat{x})(y - \hat{y})g(x)h(y) \, dx \, dy = \ldots = 0!
\]

independent variables cannot be correlated, but the reciprocal conjecture is false!!
bivariate normal distribution
bivariate normal distribution

TWO–DIMENSIONAL RANDOM VARIABLE
suppose we have a pair of RVs: $X$ and $Y$. $X$ – takes on the values: 0, 1, …, 9 and $Y$ – 1, 2, 3, 4 and 5. The data look like this:

<table>
<thead>
<tr>
<th>YX</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0129</td>
<td>0.0149</td>
<td>0.0165</td>
<td>0.0175</td>
<td>0.0178</td>
<td>0.0175</td>
<td>0.0165</td>
<td>0.0149</td>
<td>0.0129</td>
<td>0.0108</td>
</tr>
<tr>
<td>2</td>
<td>0.0188</td>
<td>0.0217</td>
<td>0.024</td>
<td>0.0254</td>
<td>0.026</td>
<td>0.0254</td>
<td>0.024</td>
<td>0.0217</td>
<td>0.0188</td>
<td>0.0157</td>
</tr>
<tr>
<td>3</td>
<td>0.0214</td>
<td>0.0246</td>
<td>0.0272</td>
<td>0.0288</td>
<td>0.0294</td>
<td>0.0288</td>
<td>0.0272</td>
<td>0.0246</td>
<td>0.0214</td>
<td>0.0178</td>
</tr>
<tr>
<td>4</td>
<td>0.0188</td>
<td>0.0217</td>
<td>0.024</td>
<td>0.0254</td>
<td>0.026</td>
<td>0.0254</td>
<td>0.024</td>
<td>0.0217</td>
<td>0.0188</td>
<td>0.0157</td>
</tr>
<tr>
<td>5</td>
<td>0.0129</td>
<td>0.0149</td>
<td>0.0165</td>
<td>0.0175</td>
<td>0.0178</td>
<td>0.0175</td>
<td>0.0165</td>
<td>0.0149</td>
<td>0.0129</td>
<td>0.0108</td>
</tr>
</tbody>
</table>

the marginal distribution $g(x)$:

| g(x) | 0.0848 | 0.0978 | 0.1082 | 0.1146 | 0.1170 | 0.1146 | 0.1082 | 0.0978 | 0.0848 | 0.0708 |

and the marginal distribution $h(y)$:

| h(y) | 0.1522 | 0.2215 | 0.2512 | 0.2215 | 0.1522 |
bivariate discrete distribution cntd.
bivariate discrete distribution cntd.

Marginal distribution of $X$

Marginal distribution of $Y$